A Chern-Simons approach to Galilean quantum gravity in 2+1 dimensions

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ABSTRACT: We define and discuss classical and quantum gravity in 2+1 dimensions in the Galilean limit. Although there are no Newtonian forces between massive objects in (2+1)-dimensional gravity, the Galilean limit is not trivial. Depending on the topology of spacetime there are typically finitely many topological degrees of freedom as well as topological interactions of Aharonov-Bohm type between massive objects. In order to capture these topological aspects we consider a two-fold central extension of the Galilei group whose Lie algebra possesses an invariant and non-degenerate inner product. Using this inner product we define Galilean gravity as a Chern-Simons theory of the doubly-extended Galilei group. The particular extension of the Galilei group we consider is a classical double of a much studied group, the extended homogeneous Galilei group, which is also often called Nappi-Witten group. We exhibit the Poisson-Lie structure of the doubly extended Galilei group, and quantise the Chern-Simons theory using a Hamiltonian approach. Many aspects of the quantum theory are determined by the quantum double of the extended homogenous Galilei group, or Galilei double for short. We study the representation theory of the Galilei double, explain how associated braid group representations account for the topological interactions in the theory, and briefly comment on an associated non-commutative Galilean spacetime.

KEYWORDS: Models of Quantum Gravity, Chern-Simons Theories, Space-Time Symmetries, Non-Commutative Geometry

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Contents

1 Introduction

The relation between the different regimes of physics and the values of the fundamental constants c, G and $\hbar$ is frequently displayed in a cube like the one shown shown in figure 1. This paper is about the corner of this cube where G and $\hbar$ have their physical values, but $c = \infty$. The corresponding physical regime is non-relativistic or Galilean quantum gravity, which has not received much attention in the literature (but see [1] for a comprehensive recent study and further references). Here we investigate this regime in a universe with two space and one time dimension. We construct a diffeomorphism invariant theory of Galilean gravity in this setting, and show how to quantise it. One motivation for this paper is to demonstrate that this can be done at all. A related motivation comes from a curiosity about the dependence of some of the conceptual and technical problems of
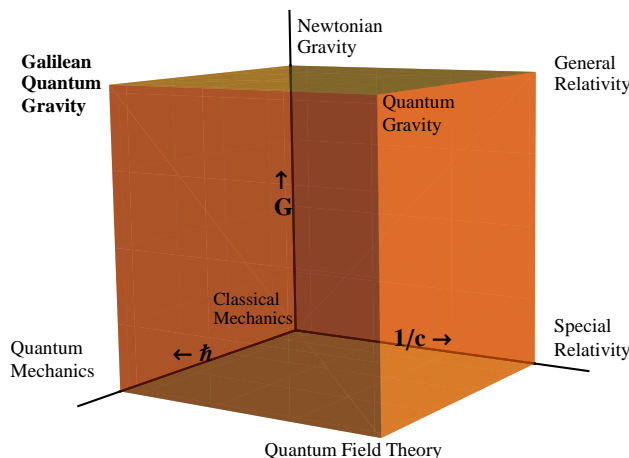
Figure 1: A cube showing the different regimes of physics, based on values of the constants c, G and $\hbar$.

quantum gravity on the speed of light. One aspect of this dependence is that Galilean gravity sits between Euclidean and Lorentzian gravity, and is therefore a good place to study the relation between the two. In the context of three-dimensional gravity, where the relative difficulty of Lorentzian to Euclidean signature is related to the non-compactness of the Lorentz group versus the compactness of the rotation group, Galilean gravity may provide a useful intermediate ground, with non-compact but commuting boost generators.

It is part of the lore of three-dimensional gravity that the Newtonian limit is trivial. If one considers the geodesic equation for a test particle in 2+1 dimensional gravity and takes the Newtonian limit in the usual way one indeed finds that there are no Newtonian forces on such a test particle [2]. However, this does not mean that the theory becomes trivial in the Galilean limit $c \to \infty$. Gravity in 2+1 dimensions has topological degrees of freedom associated with the topology of spacetime, and topological interactions between particles. An example of the latter is a deflection of a test particle in the metric of a massive particle by an amount which is related to the mass of the latter. Such interactions should survive the Galilean limit.

The question is how to capture topological interactions as $c \to \infty$. Here we approach this problem from the point of view of the Chern-Simons formulation of 2+1 dimensional gravity. In this formulation the isometry group of a model spacetime (which depends on the signature and the value of the cosmological constant) is promoted from global to local symmetry. In the case of Lorentzian signature and vanishing cosmological constant, gravity

is thus formulated as a gauge theory of the Poincaré group. One might therefore attempt to take the Galilean limit by replacing the Poincaré group with the Galilei group. However, this naive strategy fails since the invariant inner product on the Poincaré Lie algebra, which is essential for writing down the Chern-Simons action, does not have a good limit as $c \to \infty$. In fact, as we shall explain in this paper, the Lie algebra of the Galilei group in 2+1 dimensions does not possess an invariant (and non-degenerate) inner product.

In order to obtain a group of Galilean symmetries with an invariant inner product on its Lie algebra we consider central extensions of the Galilei group in two spatial dimensions. Such extensions are ubiquitous in the theoretical description of Galilei-invariant planar systems. One simple reason why central extensions are required to capture all of the physics when taking the Galilean limit is that the proportionality between rest mass and rest energy of relativistic physics is lost when $c \to \infty$. Thus, two independent generators are needed to account for mass and rest or 'internal' energy in this limit. In 3+1 dimensions, the Galilei group has only one non-trivial central extension, which is related to the mass.

In 2+1 dimensions, the Lie algebra of the Galilei group has three distinct central extensions [3–5] and the Galilei group itself has two [6–8]. In the application to non-relativistic particles, one of the central generators is related to the particle's mass and a second 'exotic' extension can be related to the non-commutativity of (suitably defined) position coordinates of the particle [9–12] but also to the particle's spin [8, 13–15]. There is considerable discussion in the literature regarding this interpretation in general and in specific models, which we shall not try to summarise here (but see e.g. [16, 17]). In this paper we adopt the spin interpretation in our notation and terminology. In particular, in our formalism the spin parameter is naturally paired with the mass parameter, making manifest an analogy between mass and spin which was stressed in [14]. Non-commutativity of position coordinates plays a role at the very end of our paper; even though the algebraic structure we obtain is similar to the one discussed in [9], the interpretation is different since our non-commutativity is controlled by universal parameters like $\hbar$ and G, and not by parameters characterising an individual particle. The third central extension of the Lie algebra does not exponentiate to the Galilei group and does not appear to have a clear physical interpretation [5]; it plays no role in this paper.

Here we show that the two-fold central extensions of the Galilei group arises naturally in a framework for taking the Galilean limit which automatically preserves the invariant inner product on the Lie algebra of the Poincaré group or, more generally, on the isometry group for the relevant value of the cosmological constant. The invariant product and the associated structures do not appear to have been studied in the literature. As a by-product of working in a wider context, we obtain central extensions with invariant inner products of the so-called Newton-Hooke groups [18, 19], which are symmetries of Galilean spacetimes with a cosmological constant. Using the inner products on the Lie algebras, one can write down Chern-Simons actions for the extended Galilei and Newton-Hooke groups. In this paper we focus on the case of vanishing cosmological constant. The Chern-Simons theory for the centrally extended Galilei group with its invariant inner product is our model for classical Galilean gravity in 2+1 dimensions. As far as we are aware, this is the first formulation of a diffeomorphism-invariant theory of Galilean gravity in 2+1 dimensions.

Interestingly, our Chern-Simons model for Galilean gravity in 2+1 dimensions turns out to be closely related to the gauge formulation of (1+1)-dimensional (or lineal) gravity [20].

The two-fold central extensions of the Galilei group and the Newton-Hooke groups all have a special property, which considerably facilitates the quantisation of the associated Chern-Simons theory: equipped with their invariant inner product, the Lie algebras of these symmetry groups all have the structure of a classical double. The double structure of the two-fold central extension of the Galilei group has a particularly simple form. The group structure is that of a semi-direct product. The homogeneous part is a central extension of the homogeneous Galilei group (the group of rotations and Galilean boosts in two spatial dimensions), which has played a role in the context of lineal gravity [20] and in the string theory literature, where it is often called Nappi-Witten group [21, 22]. The inhomogeneous (normal, abelian) part is also four-dimensional, and in duality with the Lie algebra of the homogeneous group via the invariant inner product on the Lie algebra. This is precisely the Lie-algebraic data of a classical double. Moreover, classical doubles also have a Lie co-algebra structure, which can be expressed entirely in terms of a classical r-matrix. The r-matrix endows the two-fold extension of the Galilei group (and its Newton-Hooke relatives) with the structure of a Poisson-Lie group. In the context of this paper, the existence of the classical r-matrix is important because it allows us to quantise the Chern-Simons formulation of Galilean gravity via the so-called Hamiltonian or combinatorial quantisation method [23–26].

The starting point of the combinatorial quantisation method is the construction of the Hopf algebra or quantum group which quantises the Poisson-Lie structure defined via the given r-matrix. The Hilbert space of the quantum theory as well as the action of observables and of large diffeomorphisms are then constructed in terms of the representation theory of that quantum group. In the current context, the relevant quantum group is the quantum double of the centrally extended homogeneous Galilei group, which we call the Galilei double. This quantum group is a non-co-commutative deformation of the two-fold extension of the Galilei group. In this paper we do not go through all the details of the combinatorial quantisation programme, referring the reader instead to the papers listed above and the papers [27–31] discussing its application to three-dimensional gravity. We do, however, study the Galilei double and its representation theory, and interpret some of its features in terms of Galilean quantum gravity. In particular, we show that the non-commutative addition of momenta in Galilean quantum gravity is captured by the representation ring of the quantum double, and that distance-independent, topological interactions between massive particles can be described in terms of a braid-group representation associated to the quantum double. Exploiting a generic link between non-abelian momentum manifolds and non-commutative position coordinates, we briefly comment on the non-commutative Galilean spacetime associated to the the Galilei double.

The paper is organised as follows. In section 2 we describe a unified and, to our knowledge, new framework for studying the model spacetimes of three-dimensional gravity, their isometry groups and the inner products on the corresponding Lie algebras. We use the language of Clifford algebras to obtain the two-fold extension of the Galilei Lie algebra, with its inner product, as a contraction limit of a trivial two-fold extension of the Poincaré

Lie algebra. While the contraction procedure for the Lie algebras is well-known [8], the Clifford language clarifies the origin of the central extensions and naturally provides the inner product. Similarly, we obtain the two-fold extensions of the Newton-Hooke Lie algebras as contractions of trivial central extensions of the de Sitter and anti-de Sitter Lie algebras. Section 3 contains a brief review of Galilean spacetimes and a detailed account of the doubly extended Galilei group. We explain its relation to the Nappi-Witten group, and review conjugacy classes of the latter. These conjugacy classes, already studied in [22], play an important role in many calculations in this paper. In particular, we show how to deduce coadjoint orbits of the full doubly extended Galilei group from them. These orbits, equipped with a canonical symplectic structure (which we compute), are physically interpreted as phase spaces of non-relativistic particles, possibly with spin. In section 4 we introduce the Chern-Simons theory of the doubly extended Galilei group as our model for classical Galilean gravity. We explain how to incorporate point particles by minimal coupling of the Chern-Simons actions to coadjoint orbits of the previous section. The phase space of the theory is the space of flat connections, and can be parametrised in terms of holonomies around non-contractible paths. We exhibit the classical r-matrix of the doubly extended Galilei group, and briefly explain how this r-matrix determines the Poisson structure of the phase space in the formalism of Fock and Rosly [23]. Finally, section 5 is devoted to the quantisation of Galilean gravity. We explain the role of the Galilei double in the quantisation and study its representation theory. We show how the associated braid group representation captures topological interactions between massive particles in Galilean quantum gravity, and write down the non-commutative Galilean spacetime associated to the Galilei double.

2 A unified geometrical framework for 3d gravity

The vacuum solutions of the Einstein equations in three dimensions are locally isometric to model spacetimes which depend on the signature of spacetime and on the value of the cosmological constant. As we shall explain below, each of the model spacetimes can be realised as a hypersurface in a four-dimensional embedding space with a constant metric. It turns out that these four-dimensional geometries and their Clifford algebras provide a unifying framework for discussing the isometry groups of the model spacetimes arising in 3d gravity. In particular, the possible invariant bilinear forms on the model spacetimes are naturally associated to the central elements in the even part of the Clifford algebra. For us this framework is valuable because it allows us to take the Galilean limit in such a way that we retain a non-degenerate invariant bilinear form. The resulting version of the Galilei group is naturally a two-fold central extension of the Galilei group in 2+1 dimensions.

We will use Greek indices $\mu, \nu, \ldots$ for the range $\{0, 1, 2, 3\}$ and roman indices $a, b, c, \ldots$ for $\{0, 1, 2\}$. Our convention for the spacetime metric in three dimension is $\eta_{ab} = \mathrm{diag}(-c^2, 1, 1)$. We will mostly be interested in the Lorentzian case, but the Euclidean case can be included by allowing imaginary values for c. Purely spatial indices will be denoted by Roman letters in the middle of the alphabet $i, j, k \ldots \in \{1, 2\}$. The cosmological

constant is denoted Λ here (but note that in some papers on 3d gravity, Λ stands for minus the cosmological constant in the Lorentzian case [33, 34]).

2.1 Embedding spaces, Clifford algebras and symmetries

Consider a four-dimensional auxiliary inner product space (V, g) with metric

$$g_{\mu\nu} = \text{diag}\left(-c^2, 1, 1, \frac{1}{\Lambda}\right). \tag{2.1}$$

We can define model spacetimes as embedded hypersurfaces

$$H_{c,\Lambda} = \left\{(t, x, y, w) \in \mathbb{R}^4 \,|\, -c^2 t^2 + x^2 + y^2 + \frac{1}{\Lambda} w^2 = \frac{1}{\Lambda}\right\}. \tag{2.2}$$

For finite c this gives the three-sphere S^3 ($c^2 < 0$, $\Lambda > 0$), hyperbolic space H^3 ($c^2 < 0$, $\Lambda < 0$), de Sitter space dS^3 ($c^2 > 0$, $\Lambda > 0$) and anti-de Sitter space AdS^3 ($c^2 > 0$, $\Lambda < 0$). Euclidean space E^3 and Minkowski M^3 space arise in the limit $\Lambda \to 0$, which one should take *after* multiplying the defining equation in (2.2) by Λ. In all cases the metric is induced by the embedding and has scalar curvature equal to Λ (so that $1/\sqrt{|\Lambda|}$ is the curvature radius).

We are interested in the limits $c \to \infty$ and $\Lambda \to 0$, or some combination thereof. In those limits, the inverse metric

$$g^{\mu\nu} = \text{diag}\left(-\frac{1}{c^2}, 1, 1, \Lambda\right) \tag{2.3}$$

degenerates but remains well-defined as a matrix. In studying the associated Clifford algebra (initially for finite c and non-zero Λ), we therefore work with the inverse metric and define generators γ^μ via

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}. \tag{2.4}$$

Lie algebra generators of the symmetry group $SO(V)$ of (2.3) can be realised as degree two elements in Clifford algebra

$$M^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu], \tag{2.5}$$

giving six generators, $M^{12} = \frac{1}{2}\gamma^1\gamma^2$ etc. with commutation relations

$$[M^{\kappa\lambda}, M^{\mu\nu}] = g^{\kappa\mu}M^{\lambda\nu} + g^{\lambda\nu}M^{\kappa\mu} - g^{\kappa\nu}M^{\lambda\mu} - g^{\lambda\mu}M^{\kappa\nu}. \tag{2.6}$$

We also write 1 for the identity element in the Clifford algebra, and define the volume element

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.7}$$

Then the linear span of the even powers Clifford of the Clifford generators

$$\mathcal{A} = \left[1, \gamma^5, M^{\mu\nu}\right]_{\mu,\nu=0,\dots 3} \tag{2.8}$$

is an algebra with multiplication defined by the Clifford multiplication. The elements 1 and γ^5 are central elements. Note that the element γ^5 satisfies

$$(\gamma^5)^2 = -\frac{\Lambda}{c^2} = \det(g^{\mu\nu}) \tag{2.9}$$

and acts on $\mathcal{A}$ by Clifford multiplication. As a Lie algebra with brackets defined by commutators, $\mathcal{A}$ is a trivial two-dimensional extension of the Lie algebra $so(V)$.

The elements 1 and γ^5 are, up to scale, the unique elements of, respectively, degree 1 and 4 in $\mathcal{A}$. Having chosen them, we can define maps

$$\Pi^1, \Pi^5 : \mathcal{A} \to \mathbb{R}, \tag{2.10}$$

which pick out, respectively, the coefficients of 1 and γ^5 in the expansion of an arbitrary element in $\mathcal{A}$ in any basis containing $1, \gamma^5$ and otherwise only degree two elements. Then we can define bilinear forms on $\mathcal{A}$ via

$$\langle M, N \rangle = -4\Pi^5(MN), \tag{2.11}$$

and

$$(M, N) = -4\Pi^1(MN), \tag{2.12}$$

It follows immediately from the centrality of 1 and γ^5 that these bilinear forms are invariant under the conjugation action of the spin group $Sp(V)$ (which is realised as a subset of $\mathcal{A}$). Explicitly, the pairing $\langle\ ,\ \rangle$ is non-zero whenever the indices on the basis vectors are complementary:

$$\langle M^{12}, M^{03} \rangle = -1, \quad \langle 1, \gamma^5 \rangle = -4, \quad \langle M^{12}, M^{01} \rangle = 0 \quad \text{etc.}$$

By contrast, the pairing $(\ ,\ )$ of basis vectors is non-zero whenever the indices on the basis vectors match:

$$(M^{12}, M^{12}) = 1, \quad (M^{01}, M^{01}) = -\frac{1}{c^2}, \quad (M^{13}, M^{13}) = \Lambda \quad \text{etc.}$$

The bilinear form $(\ ,\ )$ is the restriction to even elements of the usual inner product on the Clifford algebra of $(V, g^{\mu\nu})$, induced from the inner product $g^{\mu\nu}$ of degree one elements. Not surprisingly this form degenerates in both the limits $c \to \infty$ and $\Lambda \to 0$. The inner product $\langle\ ,\ \rangle$, by contrast, remains non-degenerate even when the metric $g^{\mu\nu}$ degenerates. One can also think of it of the Berezin integral in the Grassmann algebra generated by the γ^μ. The Grassmann algebra is isomorphic to the Clifford algebra as a vector space, but clearly independent of the metric $g^{\mu\nu}$. The minus signs and factors in our definition of $\langle\ ,\ \rangle$ and $(\ ,\ )$ are designed to match conventions elsewhere in the literature. The physical dimensions of the pairings follow directly from the definition: the pairing $\langle\ ,\ \rangle$ has dimensions of the inverse of γ^5 i.e length^2/time while the pairing $(\ ,\ )$ is dimensionless.

2.2 The 2+1 dimensional point of view

Next we switch notation to one that brings out the spacetime interpretation of the Lie algebra (2.6). We define the three-dimensional totally antisymmetric tensor with downstairs indices via $\epsilon_{012} = 1$. Note that

$$\epsilon^{abc} = -\frac{1}{c^2}\epsilon_{abc} \tag{2.13}$$

and that therefore

$$\epsilon_{abd}\epsilon^{aef} = -\frac{1}{c^2}\left(\delta_b^e\delta_d^f - \delta_d^e\delta_b^f\right), \qquad \epsilon_{abc}\epsilon^{dbc} = -\frac{2}{c^2}\delta_a^d. \tag{2.14}$$

Then define

$$J_a = \frac{1}{2}\epsilon_{abc}M^{bc}, \qquad P^a = M^{a3}, \tag{2.15}$$

and note that these definitions are independent of the metric $g^{\mu\nu}$. The Lie algebra brackets take the form

$$[J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, P^b] = \epsilon_a{}^b{}_c P^c, \quad [P^a, P^b] = -c^2\Lambda\epsilon^{abc}J_c. \tag{2.16}$$

Because of the identity (2.13), the right hand side of the last commutator is actually independent of c. As result, there is no mathematical difficulty in taking the Galilean limit $c \to \infty$ for the Lie algebra in terms of the generators J_a (indices downstairs) and P^a (indices upstairs). However, this would not produce the Galilei Lie algebra. To see this, we need to study the physical interpretation of the various generators. First we note the physical dimensions of the generators with indices upstairs and downstairs:

J_0: dimensionless, $\quad J^0$: 1 /velocity^2, $\qquad J_i = J^i$: 1/velocity

P_0: 1/time, $\qquad\qquad P^0$: time/length^2, $\quad P_i = P^i$: 1/length.

The Lie algebra with generators J_0, J_1, J_2 contracts to the algebra of rotations and Galilean boosts in the limit $c \to \infty$. However, since $\epsilon_i{}^0{}_j \to 0$ as $c \to \infty$, the brackets between P^0 and the J_i tends to zero in this limit; if P^0 were a time translation generator in the Galilean limit, then this bracket should give a spatial translation. Thus P^0 cannot be interpreted as a time translation generator, but P_0 can (this is confirmed by dimensional analysis). Thus, in order to take the Galilean limit we need to write the Lie brackets in terms of P_a, with lowered indices.

The brackets now read

$$[J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, P_b] = \epsilon_{abc}P^c, \quad [P_a, P_b] = -c^2\Lambda\epsilon_{abc}J^c, \tag{2.17}$$

which is the usual way of writing down the algebra of spacetime symmetry generators in 3d gravity. The pairings are

$$\langle J_a, P_b\rangle = -\eta_{ab} \tag{2.18}$$

or, explicitly,

$$\langle J_0, P_0\rangle = c^2, \qquad \langle J_i, P_j\rangle = -\delta_{ij} \tag{2.19}$$

and

$$(J_a, J_b) = -\frac{1}{c^2}\eta_{ab}, \quad (P_a, P_b) = \Lambda\eta_{ab}, \tag{2.20}$$

which is the same as

$$(J_0, J_0) = 1, \ (J_i, J_j) = -\frac{1}{c^2}\delta_{ij}, \ (P_0, P_0) = -c^2\Lambda, \ (P_i, P_j) = \Lambda\delta_{ij}. \tag{2.21}$$

Both these pairings have potential infinities in the limit $c \to \infty$, and the pairing $(\, , \,)$ has the additional problem of some entries becoming zero. In the next section we shall see that, by a suitable re-definition of generators in the bigger algebra $\mathcal{A}$, the pairing $\langle \, , \, \rangle$ remains well-defined and non-degenerate in the Galilean limit.

2.3   The Galilean limit

As explained in the introduction, the difficulties of taking the Galilean limit can be seen easily in terms of the mass-energy relation. In relativistic physics, mass and energy are proportional to each other, but the constant of proportionality tends to infinity in the Galilean limit. Thus, in order to retain generators representing mass and energy we expect the need for some kind of renormalisation when taking the speed of light to infinity.

To make contact with the conventions used in Galilean physics we first rename our generators, using the two-dimensional epsilon symbol with non-zero entries $\epsilon_{12} = -\epsilon_{21} = 1$:

$$\tilde{J} = -J_0, \qquad K_i = -\epsilon_{ij}J_j, \qquad \tilde{H} = -P_0. \tag{2.22}$$

The generator K_i, $i = 1, 2$, generate boosts in the i-th spatial direction and we will mostly use them rather than the J_i when discussing Galilean physics. Then the algebra (2.16) reads

$$[K_i, K_j] = \epsilon_{ij}\frac{1}{c^2}\tilde{J}, \qquad [K_i, \tilde{J}] = \epsilon_{ij}K_j$$
$$[K_i, P_j] = \delta_{ij}\frac{1}{c^2}\tilde{H}, \quad [K_i, \tilde{H}] = P_i, \quad [P_i, \tilde{J}] = \epsilon_{ij}P_j$$
$$[P_i, P_j] = -\epsilon_{ij}\Lambda\tilde{J}, \quad [\tilde{H}, P_i] = -c^2\Lambda K_i. \tag{2.23}$$

Now we rescale and rename the central elements in the algebra $\mathcal{A}$

$$S = \frac{1}{2c^2}\,1, \quad M = \frac{1}{2}\gamma^5. \tag{2.24}$$

As anticipated, in taking limit $c \to \infty$ we need to renormalise the energy $\tilde{H}$ and the angular momentum $\tilde{J}$ in such a way that $P^0 = \tilde{H}/c^2$ (the rest mass) and $J^0 = \tilde{J}/c^2$ (a kind of rest spin [14]) remain finite. We do this by defining

$$H = \tilde{H} - c^2M, \quad J = \tilde{J} - c^2S. \tag{2.25}$$

With this substitution, the algebra (2.23) still does not have a good limit if we take $c \to \infty$ while keeping Λ constant. The problem is the appearance of c^2 in the last line of (2.23).

If, on the other hand we let $c \to \infty$ and $\Lambda \to 0$ in such a way that $\lambda = -c^2\Lambda$ remains fixed we obtain a good limit:

$$
\begin{aligned}
&[K_i, K_j] = \epsilon_{ij}S \qquad [K_i, J] = \epsilon_{ij}K_j \\
&[K_i, P_j] = \delta_{ij}M, \quad [K_i, H] = P_i, \quad [P_i, J] = \epsilon_{ij}P_j \\
&[P_i, P_j] = \epsilon_{ij}\lambda S, \quad [H, P_i] = \lambda K_i.
\end{aligned}
\tag{2.26}
$$

This is a centrally extended version of the so-called Newton-Hooke algebra, which describes Galilean physics with a cosmological constant λ [19].

Since our re-definition of generators did not rescale the J_i it is clear that the pairing $(\ ,\ )$ degenerates in the limit $c \to \infty$. By contrast, the pairing $\langle\ ,\ \rangle$ remains non-degenerate in the Galilean limit. Using

$$
\langle S, M\rangle = -\frac{1}{c^2}, \quad \langle\tilde{H}, \tilde{J}\rangle = c^2,
\tag{2.27}
$$

we deduce

$$
\langle H, S\rangle = 1, \quad \langle M, J\rangle = 1, \quad \langle K_i, P_j\rangle = \epsilon_{ij},
\tag{2.28}
$$

or, equivalently,

$$
\langle H, S\rangle = 1, \quad \langle M, J\rangle = 1, \quad \langle J_i, P_j\rangle = -\delta_{ij},
\tag{2.29}
$$

which is an invariant inner product on the Newton-Hooke algebra, and manifestly independent of λ. The associated quadratic Casimir, which we will need later in this paper, is

$$
C_2 = H \otimes S + S \otimes H + M \otimes J + J \otimes M - (\vec{J}_i \otimes \vec{P}_i + \vec{P}_i \otimes \vec{J}_i).
\tag{2.30}
$$

Before we proceed it is worth recording the physical dimension of all the quantities introduced so far. For the generators we have the analogue of (2.2):

J: dimensionless, $\quad S$: 1 /velocity^2, $\qquad J_i = J^i$: 1/velocity

H: 1/time, $\qquad\quad M$: time/length^2, $\quad P_i = P^i$: 1/length.

It follows that the pairing $\langle\ ,\ \rangle$ is also dimensionful with dimension length^2/time. Finally, the dimension of λ is 1/time^2, so that it parametrises some kind of curvature in time.

The smallest Lie subalgebra of (2.26) containing boosts and rotations is the Lie algebra with generators J, K_1, K_2, S and brackets

$$
[K_1, K_2] = S \quad [K_i, J] = \epsilon_{ij}K_j
\tag{2.31}
$$

or, in terms of $J_i = \epsilon_{ij}K_j$,

$$
[J_1, J_2] = S \quad [J_i, J] = \epsilon_{ij}J_j.
\tag{2.32}
$$

This Lie algebra can be viewed as a central extension of the Lie algebra of the Euclidean group in two dimensions, or as the Heisenberg algebra with an outer automorphism J. In the string theory literature it is sometimes called the Nappi-Witten Lie algebra [21, 22]. In

the context of this paper it should be interpreted as central extension of the homogeneous part of the Galilei Lie algebra, so we denote it by $\hat{\mathfrak{g}}_0$. This algebra also has an invariant inner product (of Lorentzian signature!) with non-zero pairings in the above basis given by

$$\langle J, S \rangle_{\hat{\mathfrak{g}}_0} = 1, \quad \langle J_i, J_j \rangle_{\hat{\mathfrak{g}}_0} = -\delta_{ij}. \tag{2.33}$$

Note that we also have $\langle K_i, K_j \rangle_{\hat{\mathfrak{g}}_0} = -\delta_{ij}$.

One can view the Lie algebra (2.26) as a 'generalised complexification' of the homogeneous algebra $\hat{\mathfrak{g}}_0$. This viewpoint has proved useful in the context of (relativistic) three-dimensional gravity, where it was introduced for vanishing cosmological constant in [32], generalised to arbitrary values of the cosmological constant in [33] and further developed in [34]. We will often adopt it in the current paper since it provides the most efficient method for explicit calculations in many cases. The idea, explained in detail in [33], is to work with a formal parameter θ that satisfies $\theta^2 = \lambda$ and to work over the ring of numbers of the form $a + \theta b$, with $a, b \in \mathbb{R}$. The formal parameter θ is a generalisation of the imaginary unit i since it square may be negative (ordinary complex numbers), positive (hyperbolic numbers) or zero (dual numbers). In this formulation, we recover (2.26) from (2.32) by setting $H = \theta J, P_i = \theta J_i$ and $M = \theta S$. The inner product $\langle \ , \ \rangle$ (2.28) is then the 'imaginary part' (linear in θ) of the linearly extended inner product $\langle \ , \ \rangle_{\hat{\mathfrak{g}}_0}$ (2.33).

In this paper we will concentrate on the limiting situation where $\lambda = 0$. In this limit we obtain a two-fold extension of the Galilei Lie algebra, which we denote $\hat{\mathfrak{g}}$:

$$
\begin{aligned}
&[K_i, K_j] = \epsilon_{ij} S && [K_i, J] = \epsilon_{ij} K_j \\
&[K_i, P_j] = \delta_{ij} M, && [K_i, H] = P_i, && [P_i, J] = \epsilon_{ij} P_j \\
&[P_i, P_j] = 0, && [H, P_i] = 0.
\end{aligned} \tag{2.34}
$$

The associated two-fold central extension of the Galilei group, its coadjoint orbits and irreducible representations (irreps) have been studied extensively [3, 4, 8, 15]. In the next section we need to revisit some of those results in a language that, amongst others, makes use of the formal parameter θ and of the inner product (2.28). This language is designed to ease the deformation of the extended Galilei group to the Galilei quantum double in section 5.

3 The centrally extended Galilei group in two dimensions, its Lie algebra and coadjoint orbits

3.1 Galilean spacetime and symmetry

The geometrical structure of Galilean spacetimes is discussed in a number of textbooks. We base our discussion on the treatment in [35]. Thus a (2+1)-dimensional Galilean spacetime is a three-dimensional affine space A^3, with an affine map

$$t : A^3 \to \mathbb{R}, \tag{3.1}$$

which assigns to every event an absolute time. After a choice of origin we can identify A^3 with a three-dimensional vector space V^3. With the standard choice of origin for $\mathbb{R}$, the

map t then defines a linear map $V^3 \to \mathbb{R}$, which we denote by t again. The kernel of this map is the space of simultaneous events. It is a two-dimensional Euclidean vector space. We will pick a basis for V^3 consisting of a vector $\boldsymbol{e}_0$ which is normalised via $t(\boldsymbol{e}_0) = 1$ and orthonormal vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$ in the kernel of t. Then we can describe events in terms of coordinates t, x_1, x_2 relative to this basis, and identify V^3 with $\mathbb{R}^3$ (we have re-interpreted the map t as a coordinate here, but this should not lead to confusion).

In discussing $\mathbb{R}^2$ with its standard metric we write vectors as $\vec{a}, \vec{b} \dots$ and ϵ for the matrix with elements ϵ_{ij} i.e.

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.2}$$

We use the Einstein summation convention for the trivial metric δ_{ij} and introduce the abbreviations

$$\vec{a} \times \vec{b} = a_i \epsilon_{ij} b_j, \qquad \vec{a} \cdot \vec{b} = a_i b_i \qquad \vec{a}^2 = \vec{a} \cdot \vec{a}. \tag{3.3}$$

The orthochronous Galilei group is, by definition, the group of affine maps $A^3 \to A^3$ which preserves t and the Euclidean structure on the space of simultaneous events. We will mainly be interested in the identity component $G_+^\uparrow$ of the orthochronous Galilei group and its universal cover $\tilde{G}_+^\uparrow$, where we used the notation from [3]. Then $G_+^\uparrow$ consists of time- and space translations, spatial rotations and Galilean boosts. Parametrising time- and space translation in terms of $(\alpha, \vec{a}) \in \mathbb{R}^3$, rotations in terms of $\varphi \in [0, 2\pi)$ and boosts in terms of velocity vectors $\vec{v} \in \mathbb{R}^2$, we identify $G_+^\uparrow = (SO(2) \ltimes \mathbb{R}^2) \ltimes \mathbb{R}^3$, write elements in the form $(\varphi, \vec{v}; \alpha, \vec{a})$ and obtain the group multiplication law

$$(\varphi_1, \vec{v}_1; \alpha_1, \vec{a}_1)(\varphi_2, \vec{v}_2; \alpha_2, \vec{a}_2) = (\varphi_1 + \varphi_2, R(\varphi_1)\vec{v}_2 + \vec{v}_1; \alpha_1 + \alpha_2, \vec{a}_1 + R(\varphi_1)\vec{a}_2 + \alpha_2 \vec{v}_1) \tag{3.4}$$

with

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{3.5}$$

The action of the Galilei group on A^3 can be expressed in terms of the coordinate vector $(t, \vec{x}) \in \mathbb{R}^3$ as follows:

$$(\varphi, \vec{v}; \alpha, \vec{a}) : (t, \vec{x}) \mapsto (t + \alpha, R(\varphi)\vec{x} + \vec{v}t + \vec{a}). \tag{3.6}$$

The Lie algebra $\mathfrak{g}$ of $G_+^\uparrow$ is conventionally described in terms of a rotation generator J, boost generators K_1, K_2 and time- and space translation generators H, P_1, P_2. The non-vanishing brackets are

$$[K_i, H] = P_i \qquad [P_i, J] = \epsilon_{ij} P_j \qquad [K_i, J] = \epsilon_{ij} K_j. \tag{3.7}$$

It is easy to check that $P_i \otimes P_i$ and $\epsilon_{ij}(P_i \otimes K_j - K_i \otimes P_j)$ are both central elements in the universal enveloping algebra $U(\mathfrak{g})$ and that no linear combination of them is invertible i.e.

associated to an inner product on $\mathfrak{g}$. Since the dimension of the centre of $U(\mathfrak{g})$ is bounded by the rank of $\mathfrak{g}$, we conclude that the Lie algebra $\mathfrak{g}$ cannot have an invariant, non-degenerate inner product. If it did, the associated Casimir would provide a third element of the centre of $U(\mathfrak{g})$, contradicting $\mathrm{rk}(\mathfrak{g}) = 2$.

As reviewed in the Introduction, the Galilei algebra (3.7), admits three central extensions - one for each two-cocycle in the Lie algebra cohomology [3, 4]. The two-fold central extension $\hat{\mathfrak{g}}$ (2.34) we constructed via contraction in the previous section is the maximal extension of the Lie algebra that can be exponentiated to the Galilei group. We now introduce a formulation of this extension as a classical double which is particularly suitable for our purposes.

3.2 The centrally extended homogeneous Galilei group

The centrally extended homogeneous Galilei group $\hat{G}_0$ is a Lie group whose Lie algebra is the Lie algebra $\hat{\mathfrak{g}}_0$. As a manifold it is $S^1 \times \mathbb{R}^3$. Group elements can be parametrised in terms of tuples

$$(\varphi, \vec{w}, \zeta), \qquad \varphi \in [0, 2\pi), \ \vec{w} \in \mathbb{R}^2, \zeta \in \mathbb{R}, \tag{3.8}$$

which we we can identify with exponentials of the abstract generators via

$$(\varphi, \vec{w}, \zeta) \leftrightarrow \exp(\vec{w} \cdot \vec{J}) \exp(\zeta S) \exp(\varphi J). \tag{3.9}$$

The group composition law is given in [22] in complex notation and with slightly different conventions from ours (their generator J is the negative of ours, and the inner product is also the negative of ours). We choose conventions which allow us to make contact with the papers [4] and [3] on the Galilean group and its central extension. Using vector notation, the composition law can be written as

$$(\varphi_1, \vec{w}_1, \zeta_1)(\varphi_2, \vec{w}_2, \zeta_2) = (\varphi_1 + \varphi_2, \vec{w}_1 + R(\varphi_1)\vec{w}_2, \zeta), \tag{3.10}$$

where $R(\varphi)$ is an $SO(2)$ matrix given in (3.5) and

$$\zeta = \zeta_1 + \zeta_2 + \frac{1}{2}\vec{w}_1 \times R(\varphi_1)\vec{w}_2. \tag{3.11}$$

The inverse is

$$(\varphi, \vec{w}, \zeta)^{-1} = (-\varphi, -R^{-1}(\varphi)\vec{w}, -\zeta). \tag{3.12}$$

The formula for group conjugation in the group $\hat{G}_0$ plays a fundamental role in what follows. By explicit calculation, or by translating the result of [22] into our notation, one finds, in terms of

$$v = (\varphi, \vec{w}, \zeta), \qquad v_0 = (\varphi_0, \vec{w}_0, \zeta_0), \tag{3.13}$$

that

$$\mathrm{Ad}_v(v_0) = \left(\varphi_0, (1 - R(\varphi_0))\vec{w} + R(\varphi)\vec{w}_0, \zeta_0 + \frac{1}{2}(1 + R(\varphi_0))\vec{w} \times R(\varphi)\vec{w}_0 + \frac{1}{2}\sin\varphi_0 \vec{w}^2 \right). \tag{3.14}$$

We can obtain adjoint orbits by keeping only linear terms in $\varphi_0, \vec{w}_0, \zeta_0$, and using $R(\varphi_0) \approx 1 + \epsilon\varphi_0$ for small φ_0. Writing

$$\xi = \alpha J + \vec{a} \cdot \vec{J} + \tau S, \tag{3.15}$$

and with the convention

$$\vec{v} = -\epsilon\vec{w} \quad (\text{so that} \quad \vec{v} \cdot \vec{K} = \vec{w} \cdot \vec{J}) \tag{3.16}$$

one finds

$$\text{Ad}_v(\xi) = \alpha J + \vec{a}' \cdot \vec{J} + \tau' S, \tag{3.17}$$

with

$$\vec{a}' = R(\varphi)\vec{a} + \alpha\vec{v}$$
$$\tau' = \tau + \vec{v} \cdot R(\varphi)\vec{a} + \frac{1}{2}\alpha\vec{v}^2. \tag{3.18}$$

3.3 The centrally extended Galilei group

For the calculations in the rest of the paper, we need an explicit definition and description of the centrally extended Galilei group, whose Lie algebra is $\hat{\mathfrak{g}}$. We are going to base our formulation on the observation that the Lie algebra $\hat{\mathfrak{g}}$ is a 'complexification' of the Lie algebra $\hat{\mathfrak{g}}_0$ in the sense defined at the end of section 2.3. Thus we work with formal parameter θ which satisfies $\theta^2 = 0$, and use the notation (3.8) with entries of the form $a + \theta b$, with a and b real. We denote the two-fold central extension of the Galilei group by $\hat{G}$, and write elements in the form

$$(\varphi + \theta\alpha, \vec{w} + \theta\vec{a}, \zeta + \theta\tau), \qquad \alpha, \tau \in \mathbb{R}, \vec{a} \in \mathbb{R}^2, \tag{3.19}$$

with ranges for $\varphi, \vec{w}, \zeta$ as defined in (3.8). The element (3.19) can be identified with

$$\exp(\vec{w} \cdot \vec{J} + \vec{a} \cdot \vec{P}) \exp(\zeta S + \tau M) \exp(\varphi J + \alpha H). \tag{3.20}$$

This 'complexified' notation turns out be the most efficient for calculations. We use

$$R(\varphi + \theta\alpha) = R(\varphi)(1 + \epsilon\theta\alpha). \tag{3.21}$$

as well as (3.16) to compute the multiplication rule from the multiplication rule for $\hat{G}_0$:

$$(\varphi_1 + \theta\alpha_1, \vec{w}_1 + \theta\vec{a}_1, \zeta_1 + \theta\tau_1)(\varphi_2 + \theta\alpha_2, \vec{w}_2 + \theta\vec{a}_2, \zeta_2 + \theta\tau_2) \tag{3.22}$$
$$= (\varphi_1 + \varphi_2 + \theta(\alpha_1 + \alpha_2), \vec{w}_1 + R(\varphi_1)\vec{w}_2 + \theta(\vec{a}_1 + R(\varphi_1)\vec{a}_2 - \alpha_1\vec{v}_2), \zeta + \theta\tau),$$

where ζ is given as in (3.11), and

$$\tau = \tau_1 + \tau_2 + \frac{1}{2}(\vec{a}_1 \times R(\varphi_1)\vec{w}_2 + \vec{w}_1 \times R(\varphi_1)\vec{a}_2 - \alpha_1\vec{w}_1 \cdot R(\varphi_1)\vec{w}_2)$$
$$= \tau_1 + \tau_2 + \frac{1}{2}(-\vec{a}_1 \cdot R(\varphi_1)\vec{v}_2 + \vec{v}_1 \cdot R(\varphi_1)\vec{a}_2 - \alpha_1\vec{v}_1 \cdot R(\varphi_1)\vec{v}_2). \tag{3.23}$$

In studying the representation theory of $\hat{G}$ we will exploit the fact that $\hat{G}$ is a semi-direct product

$$\hat{G} = \hat{G}_0 \ltimes \mathbb{R}^4. \tag{3.24}$$

To make this manifest we write elements in factorised form

$$g = xv, \quad \text{with} \quad x = (\theta\alpha, \theta\vec{a}, \theta\tilde{\tau}), \quad v = (\varphi, \vec{w}, \zeta), \tag{3.25}$$

where we used the notation (3.19) for both elements. Note that purely inhomogeneous elements x of the group have 'purely imaginary' parameters $\theta\alpha$ etc... The factorised form can be related to the complex parametrisation (3.19) by multiplying out

$$(\theta\alpha, \theta\vec{a}, \theta\tilde{\tau})(\varphi, \vec{w}, \zeta) = (\varphi + \theta\alpha, \vec{w} + \theta(\vec{a} - \alpha\vec{v}), \zeta + \theta\tau), \tag{3.26}$$

where

$$\tau = \tilde{\tau} + \frac{1}{2}\vec{a} \times \vec{w} = \tilde{\tau} - \frac{1}{2}\vec{a} \cdot \vec{v}. \tag{3.27}$$

Using (3.18) one obtains the following multiplication law in the semi-direct product formulation

$$(\theta\alpha_1, \theta\vec{a}_1, \theta\tilde{\tau}_1)(\varphi_1, \vec{w}_1, \zeta_1)(\theta\alpha_2, \theta\vec{a}_2, \theta\tilde{\tau}_2)(\varphi_2, \vec{w}_2, \zeta_2)$$
$$= (\theta(\alpha_1 + \alpha_2), \theta(\vec{a}_1 + R(\varphi_1)\vec{a}_2 + \alpha_2\vec{v}_1), \theta\tilde{\tau})(\varphi_1 + \varphi_2, \vec{w}_1 + R(\varphi_1)\vec{w}_2, \zeta), \tag{3.28}$$

where ζ is again given by (3.11) and

$$\tilde{\tau} = \tilde{\tau}_1 + \tilde{\tau}_2 + \vec{v}_1 \cdot R(\varphi_1)\vec{a}_2 + \tfrac{1}{2}\alpha_2\vec{v}_1^2. \tag{3.29}$$

Neither the 'complexified' nor the semi-direct product description of $\hat{G}$ coincide with the parametrisation of the two-fold central extension of Galilei group most commonly used in the literature. To make contact with, for example, the parametrisation in [4] we factorise $\hat{G}$ elements in the form

$$(\theta\alpha, \vec{0}, 0)(\varphi, \vec{w} + \theta\vec{a}, \zeta + \theta\tau) = (\varphi + \theta\alpha, \vec{w} + \theta(\vec{a} - \alpha\vec{v}), \zeta + \theta\tau). \tag{3.30}$$

This almost agrees with the semi-direct product parametrisation (3.25), except for the shift (3.27). The group multiplication rule in the parametrisation (3.30) is

$$(\theta\alpha_1, \vec{0}, 0)(\varphi_1, \vec{w}_1 + \theta\vec{a}_1, \zeta_1 + \theta\tau_1)(\theta\alpha_2, \vec{0}, 0)(\varphi_2, \vec{w}_2 + \theta\vec{a}_2, \zeta_2 + \theta\tau_2) \tag{3.31}$$
$$= \big(\theta(\alpha_1 + \alpha_2), \vec{0}, 0\big)\big((\varphi_1 + \varphi_2), \vec{w}_1 + R(\varphi_1)\vec{w}_2 + \theta(\vec{a}_1 + R(\varphi_1)\vec{a}_2 + \alpha_2\vec{v}_1), \zeta + \theta\tau\big),$$

where ζ is again (3.11) and

$$\tau = \tau_1 + \tau_2 + \frac{1}{2}(\vec{a}_1 \times R(\varphi_1)\vec{w}_2 + \vec{w}_2 \times R(\varphi_1)\vec{a}_2 + \alpha_2\vec{v}_1 \times \vec{w}_2)$$
$$= \tau_1 + \tau_2 - \frac{1}{2}(\vec{a}_1 \cdot R(\varphi_1)\vec{v}_2 - \vec{v}_1 \cdot R(\varphi_1)\vec{a}_2 + \alpha_2\vec{v}_1 \cdot R(\varphi_1)\vec{v}_2). \tag{3.32}$$

This matches the conventions used for the centrally extended Galilei group in [4] up to a sign $\tau \to -\tau$.

3.4 Coadjoint orbits as adjoint orbits and the phase spaces of Galilean particles

The coadjoint orbits of the centrally extended Galilei group $\hat{G}$ were studied in detail in [8, 15]. For our purposes it is essential to identify the coadjoint orbits with adjoint orbits, using the inner product (2.29). In that way, we will be able to make contact with the group conjugacy classes which play an important role later in this paper. Thus we identify the generic element in the dual of the Lie algebra

$$Q^* = mM^* + EH^* + jJ^* + sS^* + \vec{p} \cdot \vec{P}^* + \vec{j} \cdot \vec{J}^* \tag{3.33}$$

with the Lie algebra element

$$Q = mJ + ES + jM + sH - \vec{p} \cdot \vec{J} - \vec{j} \cdot \vec{P}. \tag{3.34}$$

In order to describe adjoint orbits explicitly, we pick a reference point Q_0 in $\hat{\mathfrak{g}}$, and write a generic point in the orbit as

$$Q = \mathrm{Ad}_g(Q_0), \tag{3.35}$$

where we parametrise $g \in \hat{G}$ as is customary in the literature referring to the Galilei group (3.30):

$$g = (\theta\alpha, \vec{0}, 0)(\varphi, \vec{w} + \theta\vec{a}, \zeta + \theta\tau) = (\varphi + \theta\alpha, \vec{w} + \theta(\vec{a} - \alpha\vec{v}), \zeta + \theta\tau). \tag{3.36}$$

Before we carry out the computation we should clarify the physical interpretation of the various parameters we have introduced. A coadjoint orbit of the Galilei group (or its central extension) can be interpreted as the phase space of a Galilean particle. The coordinates on the phase space parametrise the states of motion of such a particle. A full justification of the interpretation requires Poisson structure of the phase space, which is discussed in the literature [8, 15] and which we review in section 3.5. However, we give the physical picture here so that we can interpret the formulae we derive as we go along.

The vectors $\vec{p}$ and $\vec{j}$ are the particle's momentum and impact vector (the quantity which is conserved because of invariance under Galilei boosts). The parameters m and s are the mass and spin of the particle [13], E is the energy and j the angular momentum. Note that s is related to j in the rest frame of the particle in exactly the same way in which m is related to E in that frame [14]. The interpretation of the parameters in the group element (3.36) follows from our discussion of the Galilei group in section 3.1: the vector $(\alpha, \vec{a}) \in \mathbb{R}^3$ parametrises spacetime translations, the angle φ parametrises spatial rotations, and $\vec{w}$ is related to the boost velocity via (3.16). The parameters τ and ζ do not appear to have a clear geometrical significance.

Our strategy for computing these orbits is to start with adjoint orbits of the homogeneous group $\hat{G}_0$, and to obtain orbits in the full inhomogeneous group by the complexification trick described in section 3.3. Thus we start with an element

$$Q_0 = mJ - \vec{p}_0 \cdot \vec{J} + E_0 S \tag{3.37}$$

in the Lie algebra of the homogeneous group $\hat{G}_0$. We already know the effect of conjugating with an element $v = (\varphi, \vec{w}, \zeta) \in \hat{G}_0$ from our earlier calculation (3.17). In the current notation the result reads

$$\text{Ad}_v(Q_0) = mJ - \vec{p} \cdot \vec{J} + ES, \tag{3.38}$$

with

$$\begin{aligned} \vec{p} &= R(\varphi)\vec{p}_0 - m\vec{v} \\ E &= E_0 - \vec{v} \cdot R(\varphi)\vec{p}_0 + \frac{1}{2}m\vec{v}^2. \end{aligned} \tag{3.39}$$

Physically, the formulae (3.39) give the behaviour or energy and momentum under a boost, translation and rotation of the reference frame. We will need to know the different kinds of orbits and their associated centralisers when we study the representation theory of $\hat{G}$.

Trivial orbits arise when we start with a reference element for which both mass and spin vanish:

$$Q_0(E_0) = E_0 S. \tag{3.40}$$

The orbits consists of point $O_{E_0} = \{E_0 S\}$; the associated centraliser group is the entire group i.e. $N_{E_0} = \hat{G}_0$. When the mass parameter $m \neq 0$ i.e.

$$Q_0(E_0, m) = E_0 S + mJ, \qquad m \neq 0 \tag{3.41}$$

the orbit is

$$O_{E_0, m} = \left\{ mJ - \vec{p} \cdot \vec{J} + ES \middle| E = E_0 + \frac{1}{2m}\vec{p}^2 \right\}, \tag{3.42}$$

which is a paraboloid opening towards the positive or negative E-axis, depending on the sign of m. The centraliser group is

$$N_{E_0, m} = \{v = (\varphi, \vec{0}, \zeta) \in \hat{G}_0 | \varphi \in [0, 2\pi), \zeta \in \mathbb{R}\}. \tag{3.43}$$

Finally there are non-trivial orbits when the mass is zero, but the momentum non-zero. The reference element

$$Q_0(\vec{p}_0) = -\vec{p}_0 \cdot \vec{J} \tag{3.44}$$

leads to the orbit

$$O_{\vec{p}_0} = \{ES - \vec{p} \cdot \vec{J} | |\vec{p}| = |\vec{p}_0|, E \in \mathbb{R}\}, \tag{3.45}$$

which is a cylinder. The centraliser group is

$$N_{\vec{p}_0} = \{v = (0, \lambda\vec{p}_0, \zeta) | \lambda, \zeta \in \mathbb{R}\}. \tag{3.46}$$

Moving on to the inhomogeneous group, we write a generic but fixed element in the Lie algebra $\hat{\mathfrak{g}}$ in 'complexified' language:

$$Q_0 = (m + \theta s)J - (\vec{p}_0 + \theta\vec{j}_0)\vec{J} + (E_0 + \theta j_0)S. \tag{3.47}$$

In order to compute the effect of conjugating this element with an element in $\hat{G}$ parametrised as in (3.36) we simply 'complexify' the result (3.39) i.e. we replace φ by $\varphi + \alpha\theta$, $\vec{p}_0$ by $\vec{p}_0 + \theta\vec{j}_0$ etc. Parametrising the result as

$$\text{Ad}_g(Q_0) = (m' + \theta s')J - \big(\vec{p} + \theta\vec{j}\big)\vec{J} + (E + \theta j)S \tag{3.48}$$

we see straight away that $m = m'$ and $s = s'$. The formulae $\vec{p}$ and E are simply the terms of order zero in θ, and therefore given by the expressions in (3.39). In order to find j and $\vec{j}$ we need to compute the terms which are linear in θ. Thus j is the coefficient of θ in

$$\theta j_0 + \epsilon\big(\vec{w} + \theta(\vec{a} - \alpha\vec{v})\big) \cdot R(\varphi)(1 + \epsilon\theta\alpha)(\vec{p}_0 + \theta\vec{j}_0) + \frac{1}{2}(m + \theta s)\big(\epsilon\vec{w} + \epsilon\theta(\vec{a} - \alpha\vec{v})\big)^2. \tag{3.49}$$

Multiplying out and simplifying we find

$$j = j_0 - \vec{v} \cdot R(\varphi)\vec{j}_0 + \epsilon\vec{a} \cdot R(\varphi)\vec{p}_0 + \frac{1}{2}s\vec{v}^2 + m\vec{a} \times \vec{v}. \tag{3.50}$$

In order to compute $\vec{j}$, we need the coefficient of θ in

$$R(\varphi)(1 + \epsilon\theta\alpha) \cdot (\vec{p}_0 + \theta\vec{j}_0) + (m + \theta s)\epsilon\big(\vec{w} + \theta(\vec{a} - \alpha\vec{v})\big), \tag{3.51}$$

which is

$$\vec{j} = R(\varphi)\vec{j}_0 + \alpha\epsilon R(\varphi)\vec{p}_0 + m\epsilon\big(\vec{a} - \alpha\vec{v}\big) - s\vec{v}. \tag{3.52}$$

Having obtained explicit expressions for the adjoint orbits of $\hat{\mathfrak{g}}$ it is straightforward to classify the different types of orbits which arise. This classification essentially follows from the classification of conjugacy classes in the homogeneous group $\hat{G}_0$ in [22] by linearising and using the 'complexification' trick to extend to the inhomogeneous situation. These orbits are already discussed in the literature [8, 15], albeit in a different notation. We therefore simply list the orbit types in our notation in table 1, giving in each case an element from which the orbit is obtained by conjugation. The non-trivial orbits are two- or four-dimensional. The two-dimensional orbits are essentially the adjoint orbits of $\hat{G}_0$, which return here in the inhomogenous part of $\hat{G}$. For us, the two four-dimensional orbits are most relevant: they correspond to, respectively, massive and massless particles.

3.5 Poisson structure of the orbits

The canonical one-form associated to a coadjoint $\hat{G}$-orbit labelled by one of the reference elements Q_0 in table 1 is of the general form [36]

$$\boldsymbol{\theta}_{Q_0} = \langle Q_0 \,, \, g^{-1}\mathrm{d}g\rangle. \tag{3.53}$$

To compute this, we first note the left-invariant Maurer-Cartan one-form on $\hat{G}_0$ in our parametrisation (see also [21], where this was first computed, using different coordinates). With the notation (3.13), it is

$$v^{-1}\mathrm{d}v = R(-\varphi)\mathrm{d}\vec{w} \cdot \vec{J} + \mathrm{d}\varphi J + \left(\mathrm{d}\zeta + \frac{1}{2}\mathrm{d}\vec{w} \times \vec{w}\right) S. \tag{3.54}$$

Reference element Q_0	Generic element Q on orbit	Orbit geometry
$E_0 S + j_0 M$	$E_0 S + j_0 M$	Point
$E_0 S + j_0 M + sH,$ $s \neq 0$	$E_0 S + jM + sH - \vec{j} \cdot \vec{P}$ with $\vec{j} \in \mathbb{R}^2, j = j_0 + \frac{1}{2s}\vec{j}^2$	Paraboloid$\simeq \mathbb{R}^2$
$E_0 S + j_0 M - \vec{j_0} \cdot \vec{P},$ $\vec{j_0} \neq \vec{0}$	$E_0 S + jM - \vec{j} \cdot \vec{P}$ with $j \in \mathbb{R}, |\vec{j}| = |\vec{j_0}|$	Cylinder $\simeq S^1 \times \mathbb{R}$
$E_0 S + mJ + j_0 M + sH,$ $m \neq 0$	$ES + mJ - \vec{p} \cdot \vec{J} + jM - \vec{j} \cdot \vec{P} + sH$ with $\vec{p}, \vec{j} \in \mathbb{R}^2,$ $E = E_0 + \frac{1}{2m}\vec{p}^2, j = j_0 + \frac{1}{m}\vec{p} \cdot \vec{j} - \frac{s}{2m^2}\vec{p}^2$	Tangent bundle of paraboloid$\simeq \mathbb{R}^4$
$E_0 S - \vec{p_0} \cdot \vec{J} + j_0 M - \vec{j_0} \cdot \vec{P} + sH,$ $\vec{p_0} \neq \vec{0}$	$ES - \vec{p} \cdot \vec{J} + jM - \vec{j} \cdot \vec{P} + sH$ with $E, j \in \mathbb{R}, |\vec{p}| = |\vec{p_0}|,$ $\vec{j} \cdot \vec{p} - Es = \vec{j_0} \cdot \vec{p_0} - E_0 s$	Tangent bundle of cylinder $\simeq S^1 \times \mathbb{R}^3$

Table 1: Adjoint orbits of the centrally extended Galilei group $\hat{G}$.

For later use we also record the expression for the right-invariant Maurer-Cartan form on $\hat{G}_0$

$$\mathrm{d}vv^{-1} = (\mathrm{d}\vec{w} - \mathrm{d}\varphi\epsilon\vec{w}) \cdot \vec{J} + \mathrm{d}\varphi J + \left(\mathrm{d}\zeta + \tfrac{1}{2}\vec{w}^2\mathrm{d}\varphi - \frac{1}{2}\mathrm{d}\vec{w} \times \vec{w}\right) S. \qquad (3.55)$$

Next, we again use the 'complexification' trick to obtain the left-invariant Maurer-Cartan form on the centrally extended Galilei group $\hat{G}$, which appears in (3.53). In terms of the parametrisation (3.30)

$$g = (\varphi + \theta\alpha, \vec{w} + \theta(\vec{a} - \alpha\vec{v}), \zeta + \theta\tau)$$

we find

$$\begin{aligned} g^{-1}\mathrm{d}g = {} & R(-\varphi)\mathrm{d}\vec{w} \cdot \vec{J} + \mathrm{d}\varphi J + \left(\mathrm{d}\zeta + \frac{1}{2}\mathrm{d}\vec{v} \times \vec{v}\right) S \; + R(-\varphi)\big(\mathrm{d}\vec{a} - \vec{v}\mathrm{d}\alpha\big) \cdot \vec{P} \\ & + \mathrm{d}\alpha H + \left(\mathrm{d}\tau + \frac{1}{2}\vec{v}^2\mathrm{d}\alpha + \frac{1}{2}\mathrm{d}(\vec{v} \cdot \vec{a}) - \vec{v} \cdot \mathrm{d}\vec{a}\right) M. \end{aligned} \qquad (3.56)$$

It is explained in [31], that for semi-direct product groups of the type $H \ltimes \mathfrak{h}^*$, there is an alternative formula for the symplectic structure on coadjoint orbits, which will be useful when we study Poisson-Lie group analogues of coadjoint orbits. In this alternative expression one pairs the inhomogeneous part of a generic point on the orbit with the right-invariant Maurer-Cartan form (3.55) on the homogeneous part of the group. In our case, this gives

$$\tilde{\boldsymbol{\theta}}_{Q_0} = \langle sH - \vec{j} \cdot \vec{P} + jM), \mathrm{d}vv^{-1}\rangle, \qquad (3.57)$$

where we assume the expressions for $\vec{j}$ and j given in table 1 for the generic orbit element Q on the orbit with reference element Q_0.

We briefly illustrate the formula for the symplectic potential by evaluating it for the two four-dimensional orbits in table 1. Inserting the element Q_0 representing a massive particle with spin we obtain the symplectic potential for the corresponding orbit:

$$\boldsymbol{\theta}_{E_0,j_0,m,s} = \langle E_0 S + mJ + j_0 M + sH \,, \, g^{-1}\mathrm{d}g \rangle = \tag{3.58}$$
$$= j_0\mathrm{d}\varphi + s(\mathrm{d}\zeta + \frac{1}{2}\mathrm{d}\vec{v} \times \vec{v}) + E_0\mathrm{d}\alpha + m\left(\mathrm{d}\tau + \frac{1}{2}\vec{v}^2\mathrm{d}\alpha + \frac{1}{2}\mathrm{d}(\vec{v}\cdot\vec{a}) - \vec{v}\cdot\mathrm{d}\vec{a}\right),$$

which simplifies to

$$\boldsymbol{\theta}_{E_0,j_0,m,s} = \frac{s}{2}\mathrm{d}\vec{v} \times \vec{v} + m\vec{v}\cdot\mathrm{d}(\alpha\vec{v} - \vec{a}) \quad + \text{ exact term.} \tag{3.59}$$

The associated symplectic form only depends on m and s, and is given by

$$\omega_{ms} = \mathrm{d}\boldsymbol{\theta}_{E_0,j_0,m,s} = \frac{s}{2}\mathrm{d}(\epsilon\vec{v}) \wedge \mathrm{d}\vec{v} + m\vec{v}\cdot\mathrm{d}\vec{v} \wedge \mathrm{d}\alpha - m\mathrm{d}\vec{v} \wedge \mathrm{d}\vec{a}, \tag{3.60}$$

where we use the shorthand $d\vec{a}\wedge d\vec{b} = da_i\wedge \mathrm{d}b_i$. In terms of the momentum vector $\vec{p} = -m\vec{v}$ we recover the fomula given in [13]:

$$\omega_{m,s} = \frac{s}{2m^2}\mathrm{d}(\epsilon\vec{p}) \wedge \mathrm{d}\vec{p} - \frac{1}{m}\vec{p}\cdot\mathrm{d}\vec{p} \wedge \mathrm{d}\alpha + \mathrm{d}\vec{p} \wedge \mathrm{d}\vec{a}. \tag{3.61}$$

Computing the symplectic potential via the alternative expression (3.57) we find

$$\tilde{\boldsymbol{\theta}}_{E_0,j_0,m,s} = j_0\mathrm{d}\varphi + s\left(\mathrm{d}\zeta + \frac{1}{2}\mathrm{d}\vec{v} \times \vec{v}\right) + m\left(\frac{1}{2}\vec{v}^2\mathrm{d}\alpha + \vec{a}\cdot\mathrm{d}\vec{v}\right), \tag{3.62}$$

which clearly only differs from $\boldsymbol{\theta}_{ms}$ by exact terms, and leads to the same symplectic form.

Proceeding similarly for the four-dimensional orbit corresponding to massless particles, we compute

$$\boldsymbol{\theta}_{E_0,j_0,\vec{\pi}_0,\vec{j}_0,s} = \langle E_0 S - \vec{\pi}_0 \cdot \vec{J} - \vec{j}_0 \cdot \vec{P} + j_0 M + sH \,, \, g^{-1}\mathrm{d}g \rangle \tag{3.63}$$

and find that the symplectic potential is

$$\boldsymbol{\theta}_{E_0,j_0,\vec{\pi}_0,\vec{j}_0,s} = \left(\vec{j} + \frac{s}{2}\vec{v}\right) \cdot \mathrm{d}(\epsilon\vec{v}) + \mathrm{d}\vec{p}\cdot(\alpha\vec{v} - \vec{a}) \quad + \text{ exact term,} \tag{3.64}$$

where it is understood that $\vec{p}$ and $\vec{j}$ are given by the expressions (3.39) and (3.52). Thus we obtain the symplectic potential

$$\omega_{\vec{p}_0,\vec{j}_0,s} = \mathrm{d}\vec{j} \wedge \mathrm{d}(\epsilon\vec{v}) + \frac{s}{2}\mathrm{d}\vec{v} \wedge \mathrm{d}(\epsilon\vec{v}) - \alpha\mathrm{d}\vec{p} \wedge \mathrm{d}\vec{v} - \vec{v}\cdot\mathrm{d}\vec{p} \wedge \mathrm{d}\alpha + \mathrm{d}\vec{p} \wedge \mathrm{d}\vec{a}. \tag{3.65}$$

4 Galilean gravity in three dimensions as gauge theory of the centrally extended Galilei group $\hat{G}$

4.1 The Chern-Simons formulation of Galilean gravity

Our approach to formulating a classical and, ultimately, a quantum theory of Galilean gravity is based on the Chern-Simons formulation of three-dimensional gravity discovered

in [37] and elaborated in [38] for general relativity in three dimensions. The prescription is, very simply, to pick a differentiable three-manifold $\mathcal{M}$ describing the universe and to write down the Chern-Simons action for the isometry group of the model spacetime for a given signature and value of the cosmological constant, using an appropriate inner product on the Lie algebra. Such an action is, by construction, invariant under orientation-preserving diffeomorphisms of $\mathcal{M}$. The prescription makes sense for any differentiable three-manifold $\mathcal{M}$, but for the Hamiltonian quantisation approach we need to assume $\mathcal{M}$ to be of topology $\mathbb{R} \times \Sigma$, where Σ is a two-dimensional manifold, possibly with punctures and a boundary.

For our model of Galilean gravity we use the extended Galilei group $\hat{G}$ as a gauge group, equipped with invariant inner product (2.28). The gauge field of the Chern-Simons action is then locally given by a one-form on $\mathcal{M}$ with values in the Lie algebra $\hat{\mathfrak{g}}$ of the centrally extended Galilei group:

$$A = \omega J + \omega_i K_i + \eta S + e H + e_i P_i + f M, \tag{4.1}$$

where $\omega, \omega_1, \omega_2, \eta, e, e_1, e_2$ and f are ordinary one-forms on $\mathcal{M}$. For later use we split this into a homogeneous part

$$a = \omega J + \omega_i K_i + \eta S \tag{4.2}$$

and a normal part

$$b = e H + e_i P_i + f M. \tag{4.3}$$

Then one checks that

$$\frac{1}{2}[A, A] = \epsilon_{ij}\omega_i \wedge \omega K_j + \omega_1 \wedge \omega_2 S + (\epsilon_{ij}e_i \wedge \omega + \omega_j \wedge e)P_j + \omega_i \wedge e_i M. \tag{4.4}$$

The curvature

$$F = \mathrm{d}A + \frac{1}{2}[A, A] = R + T \tag{4.5}$$

is a sum of the homogeneous terms, taking values in $\hat{\mathfrak{g}}_0$,

$$R = \mathrm{d}\omega\, J + (\mathrm{d}\eta + \omega_1 \wedge \omega_2)S + (\mathrm{d}\omega_i + \epsilon_{ij}\omega \wedge \omega_j)K_i \tag{4.6}$$

and the torsion part

$$T = \mathrm{d}e\, H + (\mathrm{d}f + \omega_i \wedge e_i)M + (\mathrm{d}e_j + \omega_j \wedge e + \epsilon_{ij}e_i \wedge \omega)P_j. \tag{4.7}$$

Thus the Chern-Simons action for Galilean gravity in 2+1 dimensions takes the form

$$
\begin{aligned}
I_{CS}[A] &= \frac{1}{16\pi G} \int_{\mathcal{M}} \langle A \wedge \mathrm{d}A \rangle + \frac{1}{3}\langle A \wedge [A, A] \rangle \\
&= \frac{1}{16\pi G} \left[\int_{\mathcal{M}} \omega \wedge \mathrm{d}f + f \wedge \mathrm{d}\omega + e \wedge \mathrm{d}\eta + \eta \wedge \mathrm{d}e + \epsilon_{ij}(\omega_i \wedge \mathrm{d}e_j + e_j \wedge \mathrm{d}\omega_i) \right. \\
&\qquad\qquad \left. + 2\int_{\mathcal{M}} \omega \wedge \omega_i \wedge e_i + e \wedge \omega_1 \wedge \omega_2 \right],
\end{aligned}
\tag{4.8}
$$

where G is Newton's constant in 2+1 dimensions, with physical dimension of inverse mass. With both the connection A and the curvature being dimensionless, the physical dimension length^2/time of the pairing $\langle , \rangle$ combines with the dimension mass of $1/G$ to produce the required dimension of an action.

As expected for gauge groups of the semi-direct product form, with a pairing between the homogenous and the normal part, this action is equivalent to an action of the BF form,

$$
\begin{aligned}
I_{BF}[A] &= \frac{1}{8\pi G} \int_{\mathcal{M}} \langle R \wedge b \rangle \\
&= \frac{1}{8\pi G} \int_{\mathcal{M}} \mathrm{d}\omega \wedge f + \mathrm{d}\eta \wedge e + \epsilon_{ij}\mathrm{d}\omega_i \wedge e_j + e \wedge \omega_1 \wedge \omega_2 + \omega \wedge \omega_i \wedge e_i, \quad (4.9)
\end{aligned}
$$

with I_{CS} and I_{BF} differing by an integral over an exact form i.e. a boundary term. This BF-action is closely related to the action of lineal gravity [20]: the two actions are based on the same curvature R and the same inner product (2.33). The difference is that the one-form b is a Lagrange multiplier *function* in the gauge formulation of lineal gravity.

In the absence of a boundary and punctures on Σ the stationary points of the Chern-Simons action are flat connections i.e. connections for which

$$
F = 0 \Leftrightarrow R = T = 0. \quad (4.10)
$$

These are the vacuum field equations in our Chern-Simons model of classical Galilean gravity. As pointed out in the introduction, we are not aware of any alternative formulation of diffeomorphism-invariant Galilean gravity in 2+1 dimensions, so in this sense the equations (4.10) or, better, the action (4.8) can be taken as a definition of classical (2+1)-dimensional Galilean gravity in the absence of sources.

The interpretation of solutions of (4.10) , i.e. flat $\hat{G}$-connections, in terms of Galilean spacetimes proceeds along the same lines as the interpretation of flat Poincaré group connections in terms of Lorentzian spacetimes, see e.g. [39–41]. The flatness condition means that the gauge field can be written in local patches as $A = g^{-1}\mathrm{d}g$ for a $\hat{G}$-valued function g in a given patch. Parametrising g in the semi-direct product form (3.25) $g = (\theta T, \theta \vec{X}, \theta \tau)(\phi, \epsilon \vec{V}, \zeta)$ the $\mathbb{R}^3$-valued function $(T, \vec{X})$ provides an embedding of the patch into our model Galilean space A^3 (which we can identify with $\mathbb{R}^3$ after a choice of origin and frame). In this way, a flat $\hat{G}$ connection translates into an identifications of open sets in $\mathcal{M}$ with open sets in A^3; overlapping open sets are connected by Galilean transformation. Note that coordinates τ and ζ referring to the central generators play no role in the geometrical interpretation. At this level, there is remarkably little difference between relativistic and Galilean gravity. In particular, even though the model space A^3 has an absolute time (up to a global shift), solutions of the Galilean vacuum field equations (4.10) need not have an absolute time. In particular, there can be time shifts when following the identifications along non-contractible paths.

A further important consequence of the field equations (4.10) is that a given infinitesimal diffeomorphism can be written as infinitesimal gauge transformation, as in relativistic 3d gravity [38]. This follows from Cartan's identity applied to the $\hat{G}$-connection A:

$$
\mathcal{L}_\xi A = \mathrm{d}\iota_\xi A + \iota_\xi \mathrm{d}A = D_A(\iota_\xi A), \quad (4.11)
$$

where we used the flatness condition $dA = -A \wedge A$ to write the infinitesimal diffeomorphism $\mathcal{L}_\xi$ as an infinitesimal gauge transformation with generator $\iota_\xi(A)$. Conversely we find, as in [38], that an infinitesimal local translation can be written as local diffeomorphisms provided the frame field $eH + e_i P_i$ is invertible. If that condition is met, local translations do not lead to (undesirable) gauge invariance in addition to local diffeomorphisms and homogeneous Galilei transformations. Questions regarding the role of non-invertible frames and the relation between global diffeomorphisms and large gauge transformations are difficult, as they are in the relativistic case, see [42] for a discussion and further references. However, unlike in the relativistic case, the issue of understanding the difference between the Chern-Simons and metric formulation does not arise since we do not have a metric theory of Galilean gravity in three dimensions.

In this paper we will not consider boundary components of Σ, although the boundary treatment in [43] includes the current situation as a special case. We will, however, couple particles to the gravitational field. This is done by marking points on the surface Σ and decorating them with coadjoint orbits of the extended Galilei group, equipped with the symplectic structures we studied in section 3.5. In order to couple the bulk action to particles we need to switch notation which makes the splitting $\mathcal{M} = \mathbb{R} \times \Sigma$ explicit. Referring the reader to [41, 43] for details, we introduce a coordinate x^0 along $\mathbb{R}$ and coordinates x^1, x^2 on Σ, and split the gauge field into

$$A = A_0 dx^0 + A_\Sigma, \tag{4.12}$$

where A_Σ is a one form on Σ (which may depend on x^0). For simplicity we consider only one particle, which is modelled by a single puncture of Σ at $\vec{x}_*$, decorated by a coadjoint, or equivalently adjoint, orbit. In the parametrisation of adjoint orbits of the centrally extended Galilei in (3.35), the group element $g \in \hat{G}$ now becomes a function of x^0. The combined field and particle action, with Q and Q_0 defined as in (3.35), takes the following form:

$$I_\tau[A_\Sigma, A_0, g] = \frac{1}{8\pi G} \int_\mathbb{R} dx^0 \int_\Sigma \langle \partial_0 A_\Sigma \wedge A_\Sigma \rangle - \int_\mathbb{R} dx^0 \langle Q_0, g^{-1} \partial_0 g \rangle \tag{4.13}$$
$$+ \int_\mathbb{R} dx^0 \int_\Sigma \langle A_0, \frac{1}{8\pi G} F_\Sigma - Q \delta^{(2)}(\vec{x} - \vec{x}_*) dx^1 \wedge dx^2 \rangle.$$

Varying with respect to the Lagrange multiplier A_0 we obtain the constraint

$$F_\Sigma(x) = 8\pi G Q \delta^{(2)}(\vec{x} - \vec{x}_*) dx^1 \wedge dx^2. \tag{4.14}$$

Variation with respect to A_Σ gives the evolution equation

$$\partial_0 A_\Sigma = d_\Sigma A_0 + [A_\Sigma, A_0]. \tag{4.15}$$

The equation obtained by varying g does not concern us here, but can be found e.g. in [43].

Together with (4.14) this means that the curvature F vanishes except at the 'worldline' of the puncture, where it is given by the constraint (4.14). In order to interpret the

constraint geometrically and physically we use the decomposition (4.5) of the curvature and parametrisation (3.34):

$$R^a_\Sigma = 8\pi G(mJ + ES - \vec{p}\cdot\vec{J})\delta^{(2)}(\vec{x} - \vec{x}_*)\mathrm{d}x^1 \wedge \mathrm{d}x^2$$
$$T^a_\Sigma = 8\pi G(sM + jH - \vec{j}\cdot\vec{P})\delta^{(2)}(\vec{x} - \vec{x}_*)\mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{4.16}$$

The constraints mean that the holonomy around the puncture is related to the Lie algebra element Q via

$$h = \exp(8\pi GQ), \tag{4.17}$$

where we have implicitly chosen a reference point where the holonomy begins and ends. We will return to the role of this reference point when we discuss the phase space of the theory further below. Since the element $Q = gQ_0g^{-1}$ lies in a fixed adjoint orbit, the holonomy has to lie in a fixed conjugacy class the element $\exp(Q_0)$. As we saw in section 3.4, the fixed element Q_0 combines the physical attributes of a point particle, like mass and spin. The holonomy h thus lies in fixed conjugacy class labelled by those physical attributes. As a preparation for discussing the phase space of the Chern-Simons theory we therefore study the geometry of conjugacy classes in $\hat{G}$ and their parametrisation in terms of physically meaningful parameters.

4.2 Conjugacy classes of the centrally extended Galilei group

We begin with a brief review of the conjugacy classes in $\hat{G}_0$ in our notation. The classification of conjugacy classes of $\hat{G}_0$ is given in [22], and follows from general formula (3.14). We would like to interpret this classification in terms of Galilean gravity. The key formula here is (4.17). Denoting the $\hat{G}_0$-valued part of the holonomy h by u we obtain

$$u = \exp(8\pi G(mJ + ES - \vec{p}\cdot\vec{J})). \tag{4.18}$$

In order to understand the relation between the adjoint orbits discussed after (3.38) and conjugacy classes in $\hat{G}_0$, we rescale the physical parameters mass, energy and momentum which label the adjoint orbits and define

$$\mu = 8\pi Gm, \quad \vec{\pi}_0 = 8\pi G\vec{p}_0, \quad \varepsilon_0 = 8\pi GE_0. \tag{4.19}$$

Then we define group elements corresponding to the special Lie algebra elements (3.41) and (3.44) via

$$u_0(\varepsilon_0) = (0, \vec{0}, \varepsilon_0), \qquad u_0(\mu, \varepsilon_0) = (\mu, \vec{0}, \varepsilon_0), \qquad u_0(\vec{\pi}_0) = (0, -\vec{\pi}_0, 0), \tag{4.20}$$

where we assume $\mu \neq 0, 2\pi, \vec{\pi}_0 \neq \vec{0}$. When parametrising the $\hat{G}_0$-conjugacy classes obtained by conjugating these elements we could either use parameters $E, \vec{p}, m$ in the Lie algebra or a group parametrisation of the form

$$u = (\mu, -\vec{\pi}, \varepsilon). \tag{4.21}$$

The two are related via

$$(\mu, -\vec{\pi}, \varepsilon) = \exp(8\pi G(mJ - \vec{p} \cdot \vec{J} + ES)). \qquad (4.22)$$

We exhibit the relationship between the two parametrisations for each type of conjugacy class separately. As we shall see, we can think of ε and $\vec{\pi}$ as energy and momentum parameters on the group, and μ as an angular mass coordinate, with range $[0, 2\pi]$. The compactification of the range of the mass from an infinite range to a finite interval is a familiar feature of three-dimensional gravity, in both the Lorentzian and Euclidean regime [44, 45].

For the trivial conjugacy classes consisting of $u_0(\varepsilon_0)$ we have simply $\varepsilon_0 = 8\pi GE$. The conjugacy class consists of a point $C_{\varepsilon_0} = \{(0, \vec{0}, \varepsilon_0)\}$; the associated centraliser group is the entire group i.e. $N_{\varepsilon_0} = \hat{G}_0$. The conjugacy class containing $u_0(\mu, \varepsilon_0)$ can be parametrised as

$$C_{\varepsilon_0, \mu} = \left\{ (\mu, -\vec{\pi}, \varepsilon) | \varepsilon = \varepsilon_0 + \frac{1}{4} \cot\left(\frac{\mu}{2}\right) \vec{\pi}^2 \right\}. \qquad (4.23)$$

Provided $\mu \neq \pi$ this a paraboloid opening towards the positive or negative ε-direction, depending on the sign of μ (non-zero by assumptions). When $\mu = \pi$ we obtain a plane in the three-dimensional $(\varepsilon, \vec{\pi})$ space. Pictures of all these conjugacy classes can be found in [22].

The planar conjugacy class is only one which looks qualitatively different from any of the adjoint orbits obtained in section 3.4. The planarity means that the energy parameter ε does not depend on the momentum $\vec{\pi}$, as confirmed by the dispersion relation in (4.23). However, this phenomenon is not as strange as it may at first appear. A similar effect occurs when one parametrises conjugacy classes in the group $SL(2, \mathbb{R})$ (which is the relativistic analogue of $\hat{G}_0$) in the form $\exp(\varepsilon J_0)\exp(-\vec{\pi} \cdot \vec{J})$, where J_0, J_1, J_2 are the usual Lorentz generators of $SL(2, \mathbb{R})$. When plotting the conjugacy classes containing the element $\exp(\mu J_0)$ in the $(\varepsilon, \vec{\pi})$ space, they curve up or down, or are horizontal depending on the value of μ, just like for $\hat{G}_0$.

The centraliser group associated to the conjugacy classes (4.23) is the group $N_{E_0, m} \simeq U(1) \times \mathbb{R}$ already encountered in the discussion of adjoint orbits (3.43). The relation (4.22) to the Lie algebra parameters is

$$\vec{\pi} = \frac{\sin\frac{\mu}{2}}{\frac{m}{2}} \vec{p}, \qquad (\varepsilon - \varepsilon_0) = \frac{\sin\mu}{m}(E - E_0). \qquad (4.24)$$

Finally, the conjugacy class containing $u_0(\vec{\pi}_0)$ is

$$C_{\vec{\pi}_0} = \{(0, \vec{\pi}, \varepsilon) | |\vec{\pi}| = |\vec{\pi}_0|, \varepsilon \in \mathbb{R}\}, \qquad (4.25)$$

which is a cylinder (see again [22] for a picture). The centraliser group is the group $N_{\vec{p}_0} \simeq \mathbb{R}^2$ of (3.46). The relation (4.22) to the Lie algebra parameters is

$$\vec{\pi} = 8\pi G\vec{p}, \qquad (\varepsilon - \varepsilon_0) = 8\pi G(E - E_0). \qquad (4.26)$$

Turning now to the conjugacy classes in the inhomogeneous group $\hat{G}$, we would like to parametrise the holonomies (4.17) in the form

$$h = u \cdot r, \qquad (4.27)$$

where $u \in \hat{G}_0$ and $r \in \hat{G}$ is entirely inside the normal abelian subgroup, i.e. can be written

$$r = (\theta\sigma, -\theta\vec{\iota}, \theta\iota), \quad \sigma, \iota \in \mathbb{R}, \vec{\iota} \in \mathbb{R}^2. \tag{4.28}$$

The reason for this parametrisation is that the Poisson structure, to be studied further below, takes a particularly simple form in these coordinates. Note that the factorisation is similar to the semi-direct product factorisation (3.25), but of the opposite order.

Writing u again as in (4.21) we multiply out to find

$$h = \left(\mu + \theta\sigma, -(\vec{\pi} + \theta R(\mu)\vec{\iota}), \varepsilon + \theta\left(\iota + \frac{1}{2}\vec{\pi} \times R(\mu)\vec{\iota}\right)\right). \tag{4.29}$$

In order find how the coordinates $\iota, \vec{\iota}$ transform under conjugation (σ is invariant), we use the following conjugation formula, with the conjugating element $g \in \hat{G}$ written in the semi-direct product parametrisation (3.25) i.e. $g = xv$:

$$\begin{aligned}
u'r' &= (xv)(ur)(xv)^{-1} \\
&= x(vuv^{-1} \, vrv^{-1})x^{-1} \\
&= (vuv^{-1}) \, (vu^{-1}v^{-1} \, x \, vuv^{-1} \, vrv^{-1} \, x^{-1}).
\end{aligned} \tag{4.30}$$

or, in a more compact form

$$u' = vuv^{-1}, \qquad r' = (u')^{-1}xu' \, x^{-1} \, vrv^{-1}, \tag{4.31}$$

where we have used the commutativity of the inhomogeneous part of $\hat{G}$. Note that the multiplication of purely inhomogeneous elements, with all parameters proportional to θ, amounts to the *addition* of the corresponding parameters.

Using the general formula (4.31) we classify the conjugacy classes of $\hat{G}$. The results are summarised in table 2, which should be seen as the group analogue of the adjoint orbits in table 1. The two-dimensional orbits are, in fact, the same as in the Lie algebra case, except that they now live in the inhomogeneous, abelian part of the group $\hat{G}$. The four-dimensional conjugacy classes are obtained by exponentiating the four-dimensional adjoint orbits into the group. Thus their topology is exactly the same as that of the corresponding adjoint orbits, but their embedding in the group, and hence their parametrisation in terms of coordinates (4.21) and (4.28) is new. In particular, the 'dispersion relations' between the group parameters defining each classes take an unfamiliar form. We list them in each case. We also give reference group elements from which the conjugacy classes can be obtained by conjugation. The parameters appearing in them are related to those in the reference elements of table 1 via (4.20) and

$$\sigma = 8\pi G\sigma, \quad \vec{\iota}_0 = 8\pi G\vec{j}, \quad \iota_0 = 8\pi Gj_0. \tag{4.32}$$

Reference element	Generic element in conjugacy class	Geometry of conjugacy class
$(0,\vec{0},\varepsilon_0)(0,\vec{0},\theta\iota_0)$	$(0,\vec{0},\varepsilon_0)(0,\vec{0},\theta\iota_0)$	Point
$(0,\vec{0},\varepsilon_0)(\theta\sigma,\vec{0},\theta\iota_0)$ $\sigma\neq 0$	$(0,\vec{0},\varepsilon_0)(\theta\sigma,-\theta\vec{\iota},\theta\iota)$ with $\vec{\iota}\in\mathbb{R}^2,\ \iota=\iota_0+\frac{1}{2\sigma}\vec{\iota}^2$	Paraboloid $\simeq\mathbb{R}^2$
$(0,\vec{0},\varepsilon_0)(0,\theta\vec{\iota_0},\theta\iota_0)$ $\vec{\iota_0}\neq\vec{0}$	$(0,\vec{0},\varepsilon_0)(0,-\theta\vec{\iota},\theta\iota)$ with $\iota\in\mathbb{R},\ |\vec{\iota}|=|\vec{\iota_0}|$	Cylinder $\simeq S^1\times\mathbb{R}$
$(\mu,\vec{0},\varepsilon_0)(\theta\sigma,\vec{0},\theta\iota_0)$ $\mu\neq 0$	$(\mu,-\vec{\pi},\varepsilon)(\theta\sigma,-\theta\vec{\iota},\theta\iota)$ with $\varepsilon=\varepsilon_0+\frac{1}{4}\cot\frac{\mu}{2}\vec{\pi}^2$, $\iota=\iota_0+\frac{\sigma\vec{\pi}^2}{8\sin^2\frac{\mu}{2}}+\frac{1}{2}(\epsilon+\cot\frac{\mu}{2})\vec{\pi}\cdot R(\mu)\vec{\iota}$	Tangent bundle of paraboloid or plane $\simeq\mathbb{R}^4$
$(0,-\vec{\pi_0},\varepsilon_0)(\theta\sigma,-\theta\vec{\iota_0},\theta\iota_0)$ $\vec{\pi_0}\neq\vec{0}$	$(0,-\vec{\pi},\varepsilon)(\theta\sigma,-\theta\vec{\iota},\theta\iota)$ with $|\vec{\pi}|=|\vec{\pi_0}|$, $\vec{\pi}\cdot\vec{\iota}-\varepsilon\sigma=\vec{\pi_0}\cdot\vec{\iota_0}-\varepsilon_0\sigma$	Tangent bundle of cylinder $\simeq S^1\times\mathbb{R}^3$

Table 2: Conjugacy classes of the centrally extended Galilei group $\hat{G}$.

4.3 The phase space of Galilean 3d gravity and the Poisson-Lie structure of $\hat{G}$

The phase space of Chern-Simons theory with gauge group H on a three-manifold of the form $\mathbb{R}\times\Sigma$ is the space of flat H connections on Σ, equipped with Atiyah-Bott symplectic structure [46, 47]. This space can be parametrised in terms of holonomies around the non-contractible loops in Σ, starting and ending at an arbitrarily chosen reference point, modulo conjugation (the residual gauge freedom at the reference point). For a very brief review of these facts in the context of 3d gravity we refer the reader to the talks [27, 48], where additional references can be found. The symplectic structure on the phase space can be described in a number of ways. The one which is ideally suited to the combinatorial quantisation approach used in this paper is a description due to Fock and Rosly [23]. The idea is to work on the extended phase space of all holonomies around non-contractible loops on Σ (without division by conjugation), and to define a Poisson bracket on the extended phase space in terms of a classical r-matrix which solves the classical Yang-Baxter equation and which is compatible with the inner product used in the Chern-Simons action. The compatibility conditions is simply that the symmetric part of the r-matrix equals the Casimir element associated with the inner product used in defining the Chern-Simons action.

The classical r-matrix endows the gauge group with a Poisson structure which is compatible with its Lie group structure, i.e. with a Poisson-Lie structure. It turns out that the Poisson structure on the extended phase space considered by Fock and Rosly can be described in terms of two standard Poisson structures associated to the Poisson-Lie group PL: one is called the Heisenberg double (a non-linear version of the cotangent bundle of PL) and the other is the dual Poisson-Lie structure (a non-linear version of the dual of the

Lie algebra of PL) [49]. The extended phase space for Chern-Simons theory on $\mathbb{R} \times \Sigma_{gn}$, where Σ_{gn} is a surface of genus g with n punctures, can then shown [49] to be isomorphic, as a Poisson manifold, to a cartesian product of g copies of the Heisenberg double of PL and a symplectic leaf of the dual Poisson Lie group for every puncture. The Fock-Rosly Poisson structure on the extended phase space for gauge groups of the semi-direct product form $PL = H \ltimes \mathfrak{h}^*$ is studied in detail in [31]. Since the group $\hat{G}$ is of this form, all of the results from [31] can be directly imported. In particular, it is shown in [31] that the Heisenberg double for groups of the form $H \ltimes \mathfrak{h}^*$ is simply the cotangent bundle of $H \times H$. The dual Poisson structure is more interesting. We now work it out in detail for the group $\hat{G}$.

The Lie algebra $\hat{\mathfrak{g}}$ has the structure of a classical double: it is the direct sum of the Lie algebras $\hat{\mathfrak{g}}_0$ (spanned by J, J_1, J_2, S) and the abelian Lie algebra spanned by H, P_1, P_2, M. These two Lie algebras are in duality via the pairing $\langle\ ,\ \rangle$, so we can identify the abelian Lie algebra with the dual vector space $\hat{\mathfrak{g}}_0^*$. Thus $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_0^*$, and the data we have just described is precisely that of a Manin triple [50]. It follows that $\hat{\mathfrak{g}}$ has, in fact, the structure of a Lie bi-algebra, which means that it has both commutators and co-commutators, with a compatibility between the two. The co-commutators can be expressed in terms of the r-matrix

$$r = M \otimes J + H \otimes S - P_i \otimes J_i, \tag{4.33}$$

which pairs generators with their dual, and is clearly compatible with the inner product $\langle\ ,\ \rangle$ in the sense of Fock and Rosly: the symmetric part of r is precisely the Casimir (2.30). On the classical double $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_0^*$ the Lie algebra structure is that of (2.34), while the co-commutator is given via

$$\delta(X) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)(r), \qquad X \in \hat{\mathfrak{g}} \tag{4.34}$$

or explicitly

$$\begin{aligned} \delta(S) = \delta(J) = 0, && \delta(J_i) = 0 \\ \delta(M) = 0,\ \delta(H) = -\epsilon_{ij} P_i \otimes P_j && \delta(P_i) = \epsilon_{ij}(M \otimes P_j - P_j \otimes M). \end{aligned} \tag{4.35}$$

As mentioned above, the r-matrix also gives rise to a Poisson structure on the group $\hat{G}$, and to associated dual and Heisenberg double Poisson structures. For explicit formulae in a more general context we refer the reader to [31]. Here we note that the symplectic leaves of the dual Poisson structure, which are associated to the punctures in the Fock-Rosly construction, can be mapped to conjugacy classes in the original group $\hat{G}$. The symplectic potentials for the symplectic form on these leaves can be expressed very conveniently in manner which is entirely analogous to the expression (3.57) for the symplectic potentials on (co)adjoint orbits. The only difference is that the inhomogeneous Lie algebra element in (3.57) is replaced by its group analogue in the factorisation (4.28), leading to the symplectic potential

$$\tilde{\boldsymbol{\theta}}_{\hat{G}^*} = \langle (\sigma H - \vec{\iota} \cdot \vec{P} + \iota M), \mathrm{d}v v^{-1} \rangle, \tag{4.36}$$

where $r = (\theta\sigma, -\theta\vec{\iota}, \theta\iota)$ is the inhomogeneous part of a group element in the given conjugacy class. By inserting the parameters $\sigma, \iota, \vec{\iota}$ for the various conjugacy classes given in table 2 one obtains the symplectic potentials on each of the conjugacy classes listed there. One can attempt to simplify the resulting expressions using (4.31), but we have not been able to generate illuminating formulae in this way.

The upshot of the Fock-Rosly construction for Chern-Simons theory with gauge group $\hat{G}$ is thus as follows. For a universe of topology $\mathbb{R} \times \Sigma_{gn}$, and assuming for definiteness that the punctures are decorated with coadjoint orbits O_{m_i,s_i}, $i = 1 \ldots n$, representing particles with masses m_i and spins s_i, the extended phase space is

$$\tilde{\mathcal{P}} = \hat{G}^{2g} \times C_{\mu_n \sigma_n} \times \ldots C_{\mu_1 \sigma_1}, \tag{4.37}$$

where we have written $C_{\mu_i \sigma_i}$ for the conjugacy classes obtained by exponentiating the orbits $O_{m_i s_i}$, and assumed that none of the μ_i equal an integer multiple of 2π. We have also suppressed the labels E_0, j_0 and their group analogues ε_0, ι_0 since they do not affect the orbit geometry and symplectic structure.

The physical phase space is the quotient

$$\begin{aligned}\mathcal{P} = \{&(A_g, B_g, \ldots, A_1, B_1, M_n, \ldots M_1) \in \tilde{\mathcal{P}}| \\ &[A_g, B_g^{-1}] \ldots [A_1, B_1^{-1}] M_n \ldots M_1 = 1\}/\text{conjugation}. \end{aligned} \tag{4.38}$$

The non-trivial part of the phase space, apart from the quotient, are the conjugacy classes in $\hat{G}$. As explained above, these are precisely the symplectic leaves of the dual Poisson-Lie group $\hat{G}^*$, with symplectic potential (4.36).

5 Towards Galilean quantum gravity in 2+1 dimensions

Our description of classical Galilean gravity in terms of a Chern-Simons theory and a compatible r-matrix is tailor-made for the application of the combinatorial quantisation programme summarised in the introduction. As explained there, all aspects of the quantisation - the construction of the Hilbert space, the representation of observables and the implementation of symmetries on the Hilbert space - are determined by a quantum group associated to the gauge group used in the Chern-Simons action and the compatible r-matrix. Since the centrally extended Galilei group with the classical r-matrix (4.33) is an example of a classical double of the form $H \ltimes \mathfrak{h}^*$, it follows from the general results of [31] that the relevant quantum group for Galilean quantum gravity is the quantum double $D(\hat{G}_0)$ of the homogeneous part of the centrally extended Galilei group or, in short, the Galilei double. The focus of this final section is therefore the definition and representation theory of $D(\hat{G}_0)$. Since this is closely related to the representation theory of centrally extended Galilei group $\hat{G}$ itself, we begin with a quick review of the latter.

5.1 Representation theory of the centrally extended Galilei group

The particular version of the centrally extended Galilei group studied in this paper is one of the central extensions of the Galilei group whose representation theory is studied in [3].

We briefly review the representation theory in our notation, with a particular emphasis on the fact that, for our version of the extended Galilei group, the representation theory easily follows from the general representation theory of semi-direct products as developed in [51] and explained, for example, in [52].

Irreducible representations of semi-direct products of form $H \ltimes N$, where N is an abelian group, are labelled by H-orbits on the dual of H (the space of characters) and irreps of centraliser groups of chosen points on those orbits. In the case at hand, where the semi-direct product is of the particular form $\hat{G}_0 \ltimes \hat{\mathfrak{g}}_0^*$, the dual of the abelian group $\hat{\mathfrak{g}}_0^*$ is the Lie algebra $\hat{\mathfrak{g}}_0$ itself (viewed as a vector space). The relevant orbits in this case are therefore the adjoint orbits of $\hat{G}_0$ which we listed, together with associated centraliser groups, in section 3.4.

As for general semi-direct products, we can describe the irreps either in terms of functions on the homogeneous group $\hat{G}_0$ obeying an equivariance condition, or in terms of functions directly on the orbits (this is rather analogous to the description of the symplectic leaves in section 3.5 either in terms of the conjugating group element or the orbit itself). The description in terms of functions on $\hat{G}_0$ has a simpler formula for the action of $\hat{G}$ elements, and easily generalises to the quantum group $\hat{G}$. However, in the literature on the Galilei group, the second view point is commonly used, and we will adopt it in this paper, too. Since the representation theory is well-documented in the literature [3, 4], we only sketch the method and then illustrate it with one case in order to establish a clear dictionary between our notation and that used, for example, in [3]

For definiteness we consider the case of a massive particle with spin. The relevant adjoint orbit of $\hat{G}_0$ is the orbit $O_{E_0,m}$ (3.42), which we can parametrise entirely in terms of the unconstrained momentum vector $\vec{p}$, with the energy given by

$$E = E_0 + \frac{1}{2m}\vec{p}^2. \tag{5.1}$$

Homogeneous elements $v \in \hat{G}_0$ act on the orbit via adjoint action Ad_v (3.38) and, by a slight abuse of notation, we write this action as $\mathrm{Ad}_v(\vec{p})$. In other words, it is understood that E changes under Galilei boosts according to (3.38), which ensure that the energy-momentum relation (5.1) is maintained. The carrier space of the irreducible representation is then simply the space $L^2(\mathbb{R}^2)$ of square-integrable functions on the space of momenta.

In order to write down the action of a general element $g \in \hat{G}$ on states $\psi \in L^2(\mathbb{R}^2)$ we need the reference point $Q_0(E_0, m)$ (3.41) on the orbit, and we need to pick a map

$$S : O_{E_0,m} \to \hat{G}_0, \tag{5.2}$$

which associates to a given point $Q = mJ - \vec{p} \cdot \vec{J} + ES \in O_{E_0,m}$ a $\hat{G}_0$-element $S(\vec{p}) = v$ satisfying

$$\mathrm{Ad}_v(Q_0) = Q, \tag{5.3}$$

i.e. v is an element of the homogeneous Galilei group which rotates/boosts the reference point Q_0 to the given point Q on the adjoint orbit. In the case at hand, we can, for

example, pick

$$S(\vec{p}) = \left(0, -\epsilon\frac{\vec{p}}{m}, 0\right), \tag{5.4}$$

where we have again used $\vec{p}$ as a coordinate on $O_{E_0,m}$. Then we define the so-called multiplier function [52]

$$n : \hat{G}_0 \times O_{E_0,m} \to N_{E_0,m}, \qquad n(v, \vec{p}) = (S(\vec{p}))^{-1}v\,S(\mathrm{Ad}_{v^{-1}}(\vec{p})). \tag{5.5}$$

The key properties of this function (both easy to check) are that it takes values in the centraliser group $N_{E_0,m}$ (3.43) associated to the orbit $O_{E_0,m}$, and that it satisfies the cocycle condition

$$n(v_1, \vec{p})n(v_2, \mathrm{Ad}_{v_1^{-1}}\vec{p}) = n(v_1v_2, \vec{p}).$$

Computing explicitly, we find, for $v = (\varphi, \vec{w}, \zeta)$, again with the convention $\vec{v} = -\epsilon\vec{w}$, that

$$n(v, \vec{p}) = \left(\varphi, \vec{0}, \zeta + \frac{1}{2m}\vec{p} \times \vec{v}\right). \tag{5.6}$$

Then, picking an irrep $\pi_{j_0,s}$ of the centraliser group $N_{E_0,m} \simeq U(1) \times \mathbb{R}$, labelled by an integer j_0 and a real number s, the action of an element $g = (\theta\alpha, \theta\vec{a}, \theta\tau)(\varphi, \vec{w}, \zeta) \in \hat{G}$ parametrised in the semi-direct product form (3.25) on a state $\psi \in L^2(\mathbb{R}^2)$ is

$$(\Pi_{E_0,m,j_0,s}(g)\,\psi)(\vec{p}) = \pi_{j_0,s}(n(v,\vec{p}))e^{i(m\tau+\vec{a}\cdot\vec{p}+E\alpha)}\,\psi(R(-\varphi)(\vec{p}-m\vec{v}))$$
$$= e^{i(j_0\varphi+s(\zeta+\frac{1}{2m}\vec{p}\times\vec{v})+m\tau+\vec{a}\cdot\vec{p}+\alpha(E_0+\frac{1}{2m}\vec{p}^2))}\,\psi(R(-\varphi)(\vec{p}-m\vec{v})). \tag{5.7}$$

In comparing this expression with representations in the literature readers should be aware of the different parametrisation (3.30) frequently used for the Galilei group, and the relation to our semi-direct parametrisation explained after (3.30).

It is not difficult to adapt this example to the representations corresponding to the cylindrical orbits (3.45). The centraliser group is $N_{\vec{p}_0} \simeq \mathbb{R} \times \mathbb{R}$, so irreps are labelled by the orbit label $\vec{p}_0$ and two real numbers s, k characterising an irrep $\mathbb{R} \times \mathbb{R}$. Finally, note that irreps of $\hat{G}$ corresponding to the vacuum orbit (3.40) are irreps of the group $\hat{G}_0$, which were studied in [53]

5.2 The Galilei double and its irreducible representations

The transition from the inhomogeneous group $\hat{G}$ to the quantum double $D(\hat{G}_0)$ can be most easily understood at the level of representations [27]. Looking at (5.7) we see that the translation part $(\theta\alpha, \theta\vec{a}, \theta\tau)$ acts on states by multiplication with the function $\exp(i(\alpha E + \vec{a} \cdot \vec{p} + m\tau))$ on momentum space $\hat{\mathfrak{g}}_0$. In the quantum double, these 'plane waves' are replaced by general functions on the *group* $\hat{G}_0$. At the purely algebraic level, this makes no difference, but the non-abelian nature of $\hat{G}_0$ makes a crucial difference for the co-product structure. The quantum double $D(\hat{G}_0)$, called Galilei double in this paper, is a non-co-commutative Hopf algebra, with a non-trivial R-matrix and ribbon element.

As an algebra, the quantum double of a group H is the semi-direct product of the group algebra of H with algebra of functions on H. It is not trivial to give a mathematically

rigorous definition of this for the case of a Lie group H, and we refer the reader to [54] where the quantum double is identified with continuous functions on $H \times H$. Further details and the application to three-dimensional quantum gravity are discussed in [55] and [27, 29]. For a full list of the Hopf algebra structure in our conventions, including antipode, co-unit and $*$-structure, we refer the reader to [31]. Here we adopt a naive approach where elements of $D(H)$ are described as a tensor product of a group element $v \in H$ and a function f on H. Such elements correspond to singular elements in $D(H)$ in the definition of [54], but they are convenient for summarising the algebra and co-algebra structure.

$$(v_1 \otimes f_1) \bullet (v_2 \otimes f_2) = v_1 v_2 \otimes (f_1 \cdot f_2 \circ \mathrm{Ad}_{v_1^{-1}})$$
$$\Delta(v \otimes f)(u_1, u_2) = v \otimes v \, f(u_1 u_2). \tag{5.8}$$

The universal R-matrix of $D(H)$ is an element of $D(H) \otimes D(H)$; in our singular notation it is

$$R = \int_H dv (e \otimes \delta_v) \otimes (v \otimes 1), \tag{5.9}$$

where e stands for the identity element of H and 1 for the function on H which is 1 everywhere.

The irreps of quantum doubles of a continuous Lie group H, classified in [54], are labelled by the H-conjugacy classes and irreps of associated centraliser groups. For the case of the extended homogeneous Galilei group $\hat{G}_0$, this is precisely the data we collected in equations (4.23) to (4.25). The study of the representation theory in [54] is in terms of equivariant functions on the group, but it is not difficult to express the irreps in terms of functions on the conjugacy classes and multipliers like (5.6). In fact, this is language employed for irreps of the double of a finite group in [56].

Consider for definiteness the irreps corresponding to the conjugacy classes $C_{\varepsilon_0, \mu}$ (4.23), with associated centraliser group $N_{E_0, m} \simeq U(1) \times \mathbb{R}$ (3.43). Analogously to the adjoint orbits $O_{E_0, m}$ the conjugacy classes can be parametrised by the rescaled momentum vectors $\vec{\pi}$, with the rescaled energy determined by

$$\varepsilon = \varepsilon_0 + \frac{1}{4} \cot\left(\frac{\mu}{2}\right) \vec{\pi}^2. \tag{5.10}$$

The carrier space of the irrep is the space of square-integrable functions on the class $C_{\varepsilon_0, \mu}$. Using the parametrisation of that orbit in terms of the momentum $\vec{\pi}$ we could again identify the carrier space with $L^2(\mathbb{R}^2)$, but this is less convenient in this case. Instead we use the parametrisation (4.21) for elements $u \in C_{\varepsilon_0, \mu}$, where the relation (5.10) is understood. With $u_0(\mu, \epsilon_0)$ as in (4.20) we have, from (3.14),

$$\mathrm{Ad}_v(u_0(\epsilon_0, \mu)) = \left(\mu, 1 - R(\mu)\vec{w}, \varepsilon_0 + \frac{1}{2}\sin^2 \mu \vec{w}^2\right). \tag{5.11}$$

The analogue of the map (5.2) is now a map

$$S : C_{\varepsilon_0, \mu} \to \hat{G}_0, \tag{5.12}$$

which associates to $u \in C_{\varepsilon 0,\mu}$ an element $v = S(u)$ so that

$$\text{Ad}_v(u_0(\epsilon_0, \mu)) = u. \tag{5.13}$$

Using (5.11) we can pick, for example,

$$S(u) = (0, (R(\mu) - 1)^{-1}\vec{\pi}, 0), \tag{5.14}$$

so that the analogue of multiplier function (5.6) is now

$$n(v, u) = \left(\varphi, \vec{0}, \zeta + \frac{1}{2}(R(\mu) - 1)^{-1}\vec{\pi} \cdot \vec{v}\right). \tag{5.15}$$

Finally we again pick an irrep $\pi_{\iota_0\sigma}$ of the centraliser group $N_{E_0,m} \simeq U(1) \times \mathbb{R}$, labelled by an integer ι_0 and a real number σ, and denote the carrier space $L^2(C_{\varepsilon 0,\mu})$ of the associated irrep of $D(\hat{G}_0)$ by $V_{\varepsilon_0,\mu,\iota_0,\sigma}$. Then singular elements $v \otimes f \in D(\hat{G}_0)$ act on $\psi \in V_{\varepsilon_0,\mu,\iota_0,\sigma}$ via

$$\begin{aligned}
(\Pi_{\varepsilon_0,\mu,\iota_0,\sigma}(v \otimes f)\,\psi)(u) &= \pi_{\iota_0,\sigma}(n(v,u))f(u)\,\psi(\text{Ad}_{v^{-1}}(u)) \\
&= e^{i(\iota_0\varphi + s(\zeta + \frac{1}{2}(R(\mu)-1)^{-1}\vec{\pi}\cdot\vec{v})}f(u)\,\psi(\text{Ad}_{v^{-1}}(u)). \tag{5.16}
\end{aligned}$$

Analogous formulae can be derived for the other conjugacy classes. For the trivial conjugacy classes C_{ε_0}, the centraliser group is the entire group $\hat{G}_0$. As in the case of the centrally extended Galilei group, the irreps of $\hat{G}_0$ again appear as special irreps of $D(\hat{G}_0)$. The irreps of $D(\hat{G}_0)$ labelled by conjugacy classes of the form $C_{\vec{\pi}_0}$ correspond to massless particles, and closely resemble the corresponding irreps of $\hat{G}$.

5.3 Aspects of Galilean quantum gravity in 2+1 dimensions

As reviewed at the beginning of this section, the combinatorial quantisation programme expresses all aspects of the quantisation of the phase space (4.38) in terms of the representation theory of $D(\hat{G}_0)$. Having studied this representation theory, we end this paper by highlighting some of the features of the quantum theory which can be derived without much further calculation. According to [31], the Hilbert space obtained by quantising the phase space (4.38) for n massive and spinning particles on a surface Σ_{gn} of genus g is

$$\mathcal{H} = \text{Inv}\left(L^2(\hat{G}_0^{2g}) \otimes \bigotimes_{i=1}^n V_{\varepsilon_{0,i},\mu_i,\iota_{0,i},\sigma_i}\right), \tag{5.17}$$

where Inv means the invariant part of the tensor product under the action of $D(\hat{G}_0)$. The general structure of this formula is not difficult to understand. Each Heisenberg double (the Poisson manifold associated to the handles, which is the cotangent bundle of $\hat{G}_0 \times \hat{G}_0$ in the our case) quantises to $L^2(\hat{G}_0 \times \hat{G}_0)$, and each conjugacy class (associated to punctures) quantises to an irrep of $D(\hat{G}_0)$. The division by conjugation that takes one from the extended phase space (4.37) to the physical phase space (4.38) is mirrored in the quantum theory by the restriction to the invariant part of the tensor product in (5.17). We refer the reader to [31] for further details, and focus on qualitative aspects here.

Fusion rules. In order to pick out the invariant part of the tensor product in (5.17) one needs to be able to decompose tensor products of representations of $D(\hat{G}_0)$ into irreps. A general method for doing this in the case of quantum doubles of compact Lie groups is explained in [57], where it is worked out in detail for $D(SU(2))$. Extending this to include the case of $D(\hat{G}_0)$ is a technical challenge, but certain aspects of the tensor product decomposition can be read off without much work.

The tensor product decomposition of products like

$$V_{\varepsilon_{0,1},\mu_1,\iota_{0,1},\sigma_1} \otimes V_{\varepsilon_{0,2},\mu_2,\iota_{0,2},\sigma_2}, \tag{5.18}$$

which occur in (5.17), corresponds physically to a set of fusion rules for particles in Galilean gravity. They tell us how to 'add' kinetic attributes like energy and momentum of two particles. In the representation theory of the Galilei double $D(\hat{G}_0)$, the conjugacy classes labelling irreps are combined by multiplying out the group elements contained in them, and sorting the products into conjugacy classes again. The underlying rule for combining energy and momentum is thus simply to multiply out the corresponding $\hat{G}_0$ elements[1]. Thus, with

$$u_1 = (\mu_1, -\vec{\pi}_1, \varepsilon_1), \quad u_2 = (\mu_2, -\vec{\pi}_2, \varepsilon_2) \tag{5.19}$$

we have

$$u_1 u_2 = \left(\mu_1 + \mu_2, -(\vec{\pi}_1 + R(\mu_1)\vec{\pi}_2), \varepsilon_1 + \varepsilon_2 + \frac{1}{2}\vec{\pi}_1 \times R(\mu_1)\vec{\pi}_2 \right). \tag{5.20}$$

This simple rule has an interesting consequence. Even if u_1 and u_2 both belong to conjugacy classes of the type $C_{E_0,\mu}$ (4.23) describing massive particles, it is possible for the product $u_1 u_2$ to belong to belong to a conjugacy class of the type $C_{\vec{\pi}_0}$ (4.25) describing massless particles. This happens generically when the sum of the particle's (rescaled) masses is 2π, which essentially means that, in physical units, the total mass equals the Planck mass. Such particles are the non-relativistic analogue of the so-called Gott pairs in relativistic 3d gravity [58]. However, unlike in the relativistic case there is no condition on the relative speed of the two particles. As long as the sum of the (rescaled) masses is 2π, the total energy and momentum is either of the vacuum type or, generically, of the massless type.

Braid interactions. The mapping class group of the spatial surface Σ_{gn} acts on the Hilbert space (5.17). If punctures are present, this means in particular that the braid group on n strands of Σ_g acts. The action of the generators of the braid group on the Hilbert space can be expressed in terms of the R-matrix (5.9) [64], and have a simple action on the $\hat{G}_0$-parts of the holonomies around any two punctures, which can again be derived from more elementary considerations, as explained in [55]. Assuming that the two punctures correspond to massive particles we denote their $\hat{G}_0$-valued energy-momenta again by u_1 and u_2 as in (5.19). In the quantum theory we can describe states of definite energy and momentum in terms of delta-functions δ_{u_1} and δ_{u_2} on the conjugacy classes

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[1]This fact can be derived more generally for Chern-Simons descriptions of particles in 2+1 dimensions from the fact that holonomies get multiplied when the underlying loops are concatenated, see e.g. [55].

corresponding to the particles' masses. Either by acting with the R-matrix (5.9) on the the tensor product $\delta_{u_1} \otimes \delta_{u_2}$ or via the elementary considerations in [55] one finds that the action of the braid group generator is

$$(u_1, u_2) \mapsto (u_1 u_2 u_1^{-1}, u_1), \tag{5.21}$$

which we evaluate, using (5.19):

$$u_1 u_2 u_1^{-1} = \left(\mu_2, -(1 - R(\mu_2))\vec{\pi}_1 - R(\mu_1)\vec{\pi}_2, \varepsilon_2 + \frac{1}{2}(1 + R(\mu_2))\vec{\pi}_1 \times R(\mu_1)\vec{\pi}_2 + \frac{1}{2}\sin\mu_2\vec{\pi}_1^2 \right). \tag{5.22}$$

The braid group action can be related to scattering of two massive particles [29], which was one of first aspects of 3d quantum gravity to be studied in detail in the limit where one particle is heavy, and the other can be treated as a test particle moving on the background spacetime created by the heavy particle [60, 61]. In usual 3d gravity, the scattering can be understood in terms of the test-particle's motion on the conical spacetime due to the heavy particle. The deficit angle of the conical spacetime is proportional to the mass of the heavy particle in Planck units, and this deficit angle controls the scattering cross section [44, 61]. We can recover aspects of this scattering from the formula (5.22). Taking the first particle to be the heavy particle, and setting its momentum to zero (so that we are in the heavy particle's rest frame), the conjugation action on the second (test) particle's momentum is simply

$$\vec{\pi}_2 \mapsto R(\mu_1)\vec{\pi}_2. \tag{5.23}$$

This is precisely the distance-independent deflection of the momentum direction by an angle proportional to the heavy particle's mass that is characteristic of scattering on a cone. It would be interesting to use the scattering formalism of [29], using the R-matrix (5.9), to investigate the non-relativistic scattering in more detail, and to compare with [44, 61].

Non-commutative geometry. The appearance of a non-commutative Lie group as a momentum space can be related to non-commutativity of spacetime coordinates using the representation theory of the quantum double [62]. The basic idea is to apply a non-abelian Fourier transform to switch from the momentum-space wave functions ψ in (5.7) to position-space representation, with a wavefunction necessarily living on a non-commutative spacetime and obeying a linear wave equation, see also [64]. Applying the reasoning of [62] and [64] to the Galilei double, where momenta are coordinates (i.e. functions) on the group $\hat{G}_0$, one expects the relevant non-commutative space algebra in this case to be the universal enveloping algebra of the Lie algebra $\hat{\mathfrak{g}}_0$, with generators now interpreted as position coordinates X_1, X_2, a time coordinate T and an additional generator R:

$$[X_i, R] = 8\pi G\hbar \, \epsilon_{ij} X_j, \qquad [X_1, X_2] = 8\pi G\hbar \, T. \tag{5.24}$$

Note that the combination $\hbar G$ has the dimension length^2/time, so cannot simply be interpreted as a Planck length or Planck time. This dimension is consistent with the interpretation of X_i as positions and of T as time.

The algebra (5.24) has a a number of interesting features. The operator representing Galilean time is a central element, and in that sense Galilean time remains absolute in the non-commutative regime. It is also worth noting that the algebra of spacetime coordinates closes, and is simply the universal enveloping algebra of the Heisenberg algebra. For fixed time, the non-commutativity of the space coordinates is of Moyal-type, with the deformation matrix $\Theta_{ij} = \epsilon_{ij} 8\pi G\hbar t$ proportional to time t (an eigenvalue of T). Algebraically, this is precisely the non-commutative position algebra which was studied in the context of planar Galilean physics in [9–12]. However, the status here is different. Our non-commutativity is a quantum gravity effect (proportional to both G and $\hbar$) and thus universal. The non-commutativity of position coordinates of a particle in [9] is related to the properties of a particle moving in a classical and commutative plane. Presumably the two effects would have to be somehow combined if such a particle was moving in the non-commutative spacetime (5.24). It would clearly be interesting to study the spacetime (5.24) and its physical implications in detail. The role of the generator R (the only compact generator in $\hat{\mathfrak{g}}_0$) is not clear to us. Its appearance reminiscent of the appearance of a fourth dimension in the study of the differential calculus [63] on the relativistic non-commutative spacetimes with brackets $[X_a, X_b] = \epsilon_{abc} X^c$.

Acknowledgments

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